Torus Knots and the Chern-Simons path integral: a rigorous treatment

January 28, 2016

Atle Hahn

Group of Mathematical Physics, University of Lisbon
Av. Prof. Gama Pinto, 2
PT-1649-003 Lisboa, Portugal
Email: atle.hahn@gmx.de

Abstract

In 1993 Rosso and Jones computed for every simple, complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and every colored torus knot in S^3 the value of the corresponding $U_q(\mathfrak{g}_{\mathbb{C}})$-quantum invariant by using the machinery of quantum groups. In the present paper we derive a $S^2 \times S^1$-analogue of the Rosso-Jones formula (for colored torus ribbon knots) directly from a rigorous realization of the corresponding (gauge fixed) Chern-Simons path integral. In order to compare the explicit expressions obtained for torus knots in $S^2 \times S^1$ with those for torus knots in S^3 one can perform a suitable surgery operation. By doing so we verify that the original Rosso-Jones formula is indeed recovered for every $\mathfrak{g}_{\mathbb{C}}$.

1 Introduction

Let $\mathfrak{g}_{\mathbb{C}}$ be an arbitrary simple complex Lie algebra, let $q \in \mathbb{C}\backslash\{0\}$ be either generic or a root of unity of sufficiently high order, and let $U_q(\mathfrak{g}_{\mathbb{C}})$ be the corresponding quantum group. In [35] an explicit formula for the values of the $U_q(\mathfrak{g}_{\mathbb{C}})$-quantum invariant of an arbitrary colored torus knot in S^3 was found and proven using the representation theory of $U_q(\mathfrak{g}_{\mathbb{C}})$.

In the special case where q is a root of unity the quantum invariants studied in [35] are normalized versions of the Reshetikhin-Turaev invariants associated to $M = S^3$ and $U_q(\mathfrak{g}_{\mathbb{C}})$ (cf. Eq. (A.1) in the Appendix). It is widely believed that the Reshetikhin-Turaev invariants associated to a closed oriented 3-manifold M and to $U_q(\mathfrak{g}_{\mathbb{C}})$ are equivalent to Witten's heuristic path integral expressions based on the Chern-Simons action function associated to (M, G, k) where G is the simply connected, compact Lie group corresponding to the compact real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$ and $k \in \mathbb{N}$ is chosen suitably (cf. Remark 5.6 below). Accordingly, it is natural to ask whether it is possible to derive the Rosso-Jones formula (or analogues/generalizations for base manifolds M other than S^3) directly from Witten's path integral expressions.

In the present paper I will show how one can do this for a large class of colored torus (ribbon) knots L in the manifold $M = S^2 \times S^1$ in a rigorous way. The approach of the present paper is based on the so-called torus gauge fixing procedure which was introduced in [7, 8] for the study of the CS path integral on manifolds of the form $M = \Sigma \times S^1$. In [17, 18] the basic heuristic formula of [7] was generalized to general colored links L in M. The generalized formula in [17, 18] was recently simplified in [19], cf. the heuristic equation (2.7) below, which will be the starting point for the rigorous treatment of the present paper.

In order to make rigorous sense of the RHS of the aforementioned Eq. (2.7) we will work within the simplicial setting developed in [19]. The simplicial setting not only allows a completely rigorous treatment but also one that is essentially elementary: apart from some[1] basic results from general Lie theory only a few quite simple results on oscillatory Gauss-type integrals on Euclidean vector spaces will be needed, cf. Sec. 4 below.

The paper is organized as follows:

In Sec. 2 we first recall the aforementioned heuristic formula Eq. (2.7) for the CS path integral in the torus gauge and later give a ribbon version of Eq. (2.7), cf. Eq. (2.17) below.

In Sec. 3 we introduce a (rigorous) simplicial realization $\mathrm{WLO}^{disc}_{rig}(L)$ of the RHS of the heuristic formula Eq. (2.17) in Sec. 2 for generic colored ribbon links L. The definition of $\mathrm{WLO}^{disc}_{rig}(L)$ is similar to the one in [19] but incorporates some improvements and simplifications.

In Sec. 4 we recall the relevant results from [20] on oscillatory Gauss-type integrals on Euclidean vector spaces which we will use in Sec. 5.

In Sec. 5 we compute $\mathrm{WLO}^{disc}_{rig}(L)$ (and the normalized version $\mathrm{WLO}^{disc}_{norm}(L)$) explicitly for a large class of colored torus ribbon knots L in $S^2 \times S^1$, see Theorem 5.7 and its proof. Apart from Theorem 5.7 a straightforward generalization is proven, cf. Theorem 5.8.

In Sec. 6 we combine Theorem 5.8 with a suitable surgery argument. The explicit expressions obtained in this way are then compared with those in the Rosso-Jones formula for colored torus knots in S^3. We find agreement for all $\mathfrak{g}_{\mathbb{C}}$.

The paper concludes with Sec. 7 and a short appendix.

Comment 1 The present paper pursues two closely related but still quite different goals:

Goal 1 (= Main Goal): Make progress with the simplicial program for Chern-Simons theory, cf. Sec. 3 in [19] and Remark 3.10 below.

Goal 1 is achieved by Theorem 5.7, Theorem 5.8, and the partial verification of Conjecture 1 given in Sec. 6.2.

Goal 2: Give a (new) heuristic derivation of the original Rosso-Jones formula for general $\mathfrak{g}_{\mathbb{C}}$.

Goal 2 is achieved by combining Theorem 5.8 (and Remark 3.11) below with the modification of Sec. 6.2 which is obtained by rewriting Sec. 6.2 using Witten's heuristic surgery argument instead of the (rigorous) surgery argument for the Reshetikhin-Tureav invariant, cf. Remark 6.2 below. [Of course, if we were only interested in Goal 2 and not in Goal 1 we could significantly reduce the amount of work and simply state and "prove" a heuristic continuum version of Theorem 5.8. This would take only a few pages. In particular, Sec. 3 could then be omitted.]

Regarding Goal 2 it should be noted that there are already several quite general heuristic approaches for calculating the CS path integral expressions for a large class of knots & links and base manifolds M, cf. the perturbative approach in [14, 4, 25, 2, 3, 9] based on Lorentz gauge fixing and the approach in [5, 6] which is based on non-Abelian localization. It is expected that in the special case of torus knots in $M = S^3$ the approach in [5, 6] leads to the Rosso-Jones formula but to my knowledge this has so far only been shown explicitly in the special case $\mathfrak{g}_{\mathbb{C}} = sl(2, \mathbb{C})$.

Apart from these (heuristic) path integral approaches one should also mention the approach in [26, 22, 27, 28, 29, 39] where Witten's CS path integral expressions are evaluated for torus knots & links using the heuristic "knot operator" approach introduced in [30]. The knot operator approach allows the derivation of the Rosso-Jones formula for arbitrary simple complex Lie

[1] In fact, even most of the Lie theoretic results appear only after the path integral expressions have already been evaluated explicitly (cf. Steps 1–4 in the proof of Theorem 5.7) and we compare the explicit expressions with those in the Rosso-Jones formuala (cf. Step 5 in the proof of Theorem 5.7 and Sec. 6.2)

algebras but involves only few genuine path integral arguments (i.e. arguments which deal directly/explicitly with the CS path integral). Instead, a variety of several quite different heuristic arguments, some of them from Conformal Field Theory, are used in the knot operator approach.

2 The heuristic Chern-Simons path integral in the torus gauge

Let G be a simple, simply-connected, compact Lie group and T a maximal torus of G. By $\mathfrak{g}$ and $\mathfrak{t}$ we will denote the Lie algebras of G and T and by $\langle \cdot, \cdot \rangle$ the unique Ad-invariant scalar product on $\mathfrak{g}$ such that $\langle \check{\alpha}, \check{\alpha} \rangle = 2$ for every short coroot $\check{\alpha} \in \mathfrak{t}$.

Let M be a compact oriented 3-manifold of the form $M = \Sigma \times S^1$ where Σ is a compact oriented surface. (From Sec. 5 on we will only consider the special case $\Sigma = S^2$.) Finally, let L be a fixed (ordered and oriented) "link" in M, i.e. a finite tuple $(l_1, \ldots, l_m)$, $m \in \mathbb{N}$, of pairwise non-intersecting knots l_i . We equip each l_i with a "color", i.e. an irreducible, finite-dimensional, complex representation ρ_i of G. Recall that a "knot" in M is an embedding $l : S^1 \to M$. Using the surjection $[0,1] \ni t \mapsto e^{2\pi i t} \in \{z \in \mathbb{C} \mid |z| = 1\} \cong S^1$ we can consider each knot as a loop $l : [0,1] \to M$, $l(0) = l(1)$, in the obvious way.

2.1 Basic spaces

As in [19, 20] we will use the following notation[2]

$$\mathcal{B} = C^\infty(\Sigma, \mathfrak{t}) \tag{2.1a}$$
$$\mathcal{A} = \Omega^1(M, \mathfrak{g}) \tag{2.1b}$$
$$\mathcal{A}_\Sigma = \Omega^1(\Sigma, \mathfrak{g}) \tag{2.1c}$$
$$\mathcal{A}_{\Sigma,\mathfrak{t}} = \Omega^1(\Sigma, \mathfrak{t}), \quad \mathcal{A}_{\Sigma,\mathfrak{k}} = \Omega^1(\Sigma, \mathfrak{k}) \tag{2.1d}$$
$$\mathcal{A}^\perp = \{A \in \mathcal{A} \mid A(\partial/\partial t) = 0\} \tag{2.1e}$$
$$\check{\mathcal{A}}^\perp = \{A^\perp \in \mathcal{A}^\perp \mid \int A^\perp(t) dt \in \mathcal{A}_{\Sigma,\mathfrak{k}}\} \tag{2.1f}$$
$$\mathcal{A}^\perp_c = \{A^\perp \in \mathcal{A}^\perp \mid \ A^\perp \text{ is constant and } \mathcal{A}_{\Sigma,\mathfrak{t}}\text{-valued}\} \tag{2.1g}$$

where $\mathfrak{k}$ is the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{g}$ w.r.t. $\langle \cdot, \cdot \rangle$. Above dt denotes the normalized translation-invariant volume form on S^1 and $\partial/\partial t$ the vector field on $M = \Sigma \times S^1$ obtained by "lifting" in the obvious way the normalized translation-invariant vector field $\partial/\partial t$ on S^1. In Eqs. (2.1f) and (2.1g) we used the "obvious" identification (cf. Sec. 2.3.1 in [19])

$$\mathcal{A}^\perp \cong C^\infty(S^1, \mathcal{A}_\Sigma) \tag{2.2}$$

where $C^\infty(S^1, \mathcal{A}_\Sigma)$ is the space of maps $f : S^1 \to \mathcal{A}_\Sigma$ which are "smooth" in the sense that $\Sigma \times S^1 \ni (\sigma, t) \mapsto (f(t))(X_\sigma) \in \mathfrak{g}$ is smooth for every smooth vector field X on Σ. Note that we have

$$\mathcal{A}^\perp = \check{\mathcal{A}}^\perp \oplus \mathcal{A}^\perp_c \tag{2.3}$$

2.2 The original Chern-Simons path integral

The Chern-Simons action function $S_{CS} : \mathcal{A} \to \mathbb{R}$ associated to M, G, and the "level"[3] $k \in \mathbb{N}$ is given by

$$S_{CS}(A) = -k\pi \int_M \langle A \wedge dA \rangle + \tfrac{1}{3} \langle A \wedge [A \wedge A] \rangle, \quad A \in \mathcal{A} \tag{2.4}$$

[2] Here $\Omega^p(N, V)$ denotes the space of V-valued p-forms on a smooth manifold N

[3] cf. Remark 5.6 below

Here $[\cdot \wedge \cdot]$ denotes the wedge product associated to the Lie bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and $\langle \cdot \wedge \cdot \rangle$ the wedge product associated to the scalar product $\langle \cdot,\cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$.

The (expectation value of the) "Wilson loop observable" associated to the colored link $L = (l_1, l_2, \ldots, l_m)$ fixed above is the informal integral expression given by

$$\mathrm{WLO}(L) := \int_{\mathcal{A}} \Big(\prod\nolimits_{i=1}^{m} \mathrm{Tr}_{\rho_i}(\mathrm{Hol}_{l_i}(A)) \Big) \exp(iS_{CS}(A)) DA \tag{2.5}$$

where $\mathrm{Hol}_l(A) \in G$ is the holonomy of $A \in \mathcal{A}$ around the loop $l = l_i$, $i \leq m$, and DA is the (ill-defined) "Lebesgue measure" on the infinite-dimensional space $\mathcal{A}$. A useful explicit formula for $\mathrm{Hol}_l(A)$ is

$$\mathrm{Hol}_l(A) = \lim_{n \to \infty} \prod\nolimits_{j=1}^{n} \exp\big(\tfrac{1}{n} A(l'(t))\big)_{|t=j/n} \tag{2.6}$$

where $\exp : \mathfrak{g} \to G$ is the exponential map of G.

Remark 2.1 *In the physics literature the notation* $Z(M, L)$ *and* $P\exp(\int_l A)$ *is usually used instead of* $\mathrm{WLO}(L)$ *and* $\mathrm{Hol}_l(A)$*.*

2.3 The torus gauge fixed Chern-Simons path integral

Let $\pi_\Sigma : \Sigma \times S^1 \to \Sigma$ be the canonical projection. For each loop l_i appearing in the link L we set $l^i_\Sigma := \pi_\Sigma \circ l_i$. Moreover, we fix $\sigma_0 \in \Sigma$ such that

$$\sigma_0 \notin \bigcup\nolimits_i \mathrm{arc}(l^i_\Sigma)$$

By applying "abstract torus gauge fixing" (cf. Sec. 2.2.4 in [19]) and suitable change of variable (cf. Sec. 2.3.1 and Appendix B.3 in [19]) one can derive at a heuristic level (cf. Eq. (2.53) in [19])

$$\begin{aligned}\mathrm{WLO}(L) \sim \sum\nolimits_{y \in I} \int_{\mathcal{A}_c^\perp \times \mathcal{B}} \Big\{ & 1_{C^\infty(\Sigma, \mathfrak{t}_{reg})}(B)\, \mathrm{Det}_{FP}(B) \\ & \times \Big[\int_{\check{\mathcal{A}}^\perp} \Big(\prod\nolimits_i \mathrm{Tr}_{\rho_i}\big(\mathrm{Hol}_{l_i}(\check{A}^\perp + A_c^\perp, B)\big) \Big) \exp(iS_{CS}(\check{A}^\perp, B)) D\check{A}^\perp \Big] \\ & \times \exp\big(-2\pi i k \langle y, B(\sigma_0) \rangle\big) \Big\} \exp(iS_{CS}(A_c^\perp, B))(DA_c^\perp \otimes DB)\end{aligned} \tag{2.7}$$

where "$\sim$" denotes equality up to a multiplicative "constant"[4] C, where $I := \ker(\exp_{|\mathfrak{t}}) \subset \mathfrak{t}$, where DB and $DA_c^\perp$ are the informal "Lebesgue measures" on the infinite-dimensional spaces $\mathcal{B}$ and $\mathcal{A}_c^\perp$, and where we have set $\mathfrak{t}_{reg} := \exp^{-1}(T_{reg})$, T_{reg} being the set of "regular" elements[5] of T. Moreover, we have set for each $B \in \mathcal{B}$, $A^\perp \in \mathcal{A}^\perp$

$$S_{CS}(A^\perp, B) := S_{CS}(A^\perp + Bdt), \tag{2.8}$$

$$\begin{aligned}\mathrm{Hol}_l(A^\perp, B) &:= \mathrm{Hol}_l(A^\perp + Bdt) \\ &= \lim_{n \to \infty} \prod\nolimits_{j=1}^{n} \exp\big(\tfrac{1}{n}[A^\perp(l_{S^1}(t))(l'_\Sigma(t)) + B(l_\Sigma(t)) \cdot dt(l'_{S^1}(t))]\big)_{t=j/n}\end{aligned} \tag{2.9}$$

where dt is the real-valued 1-form on $M = \Sigma \times S^1$ obtained by pulling back the 1-form dt on S^1 by means of the canonical projection $\pi_{S^1} : \Sigma \times S^1 \to S^1$ and where $l_{S^1} : [0,1] \to S^1$ and $l_\Sigma : [0,1] \to \Sigma$ are the projected loops given by $l_{S^1} := \pi_{S^1} \circ l$ and $l_\Sigma := \pi_\Sigma \circ l$.

Finally, the expression $\mathrm{Det}_{FP}(B)$ in Eq. (2.7) is the informal expression given by

$$\mathrm{Det}_{FP}(B) := \det\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B))_{|\mathfrak{k}}\big) \tag{2.10}$$

[4] "constant" in the sense that C does not depend on L. By contrast, C may depend on G, Σ, and k.

[5] i.e. the set of all $t \in T$ which are not contained in a different maximal torus T'

where $1_{\mathfrak{k}} - \exp(\mathrm{ad}(B))_{|\mathfrak{k}}$ is the linear operator on $C^\infty(\Sigma, \mathfrak{k})$ given by

$$(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B))_{|\mathfrak{k}} \cdot f)(\sigma) = (1_{\mathfrak{k}} - \exp(\mathrm{ad}(B(\sigma)))_{|\mathfrak{k}}) \cdot f(\sigma) \qquad \forall \sigma \in \Sigma, \quad \forall f \in C^\infty(\Sigma, \mathfrak{k}) \tag{2.11}$$

where on the RHS $1_{\mathfrak{k}}$ is the identity on $\mathfrak{k}$.

It will be convenient to generalize the definition of $1_{\mathfrak{k}} - \exp(\mathrm{ad}(B))_{|\mathfrak{k}}$ above. For every $p \in \{0, 1, 2\}$ we define $(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B))_{|\mathfrak{k}})^{(p)}$ to be the linear operator on $\Omega^p(\Sigma, \mathfrak{k})$ given by

$$\begin{gathered}\forall \alpha \in \Omega^p(\Sigma, \mathfrak{k}) : \forall \sigma \in \Sigma : \forall X_\sigma \in \wedge^p T_\sigma \Sigma : \\ \big((1_{\mathfrak{k}} - \exp(\mathrm{ad}(B))_{|\mathfrak{k}})^{(p)} \cdot \alpha\big)(X_\sigma) = (1_{\mathfrak{k}} - \exp(\mathrm{ad}(B(\sigma))_{|\mathfrak{k}}) \cdot \alpha(X_\sigma)\end{gathered} \tag{2.12}$$

Note that under the identification $C^\infty(\Sigma, \mathfrak{k}) \cong \Omega^0(\Sigma, \mathfrak{k})$ the operator $(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B))_{|\mathfrak{k}})^{(0)}$ coincides with what above we call $1_{\mathfrak{k}} - \exp(\mathrm{ad}(B))_{|\mathfrak{k}}$.

As in [19] we will now fix an auxiliary Riemannian metric $\mathbf{g}_\Sigma$ on Σ. Let $\ll \cdot, \cdot \gg_{\mathcal{A}_\Sigma}$ and $\ll \cdot, \cdot \gg_{\mathcal{A}^\perp}$ be the scalar products on $\mathcal{A}_\Sigma$ and $\mathcal{A}^\perp \cong C^\infty(S^1, \mathcal{A}_\Sigma)$ induced by $\mathbf{g}_\Sigma$, and let $\star : \mathcal{A}_\Sigma \to \mathcal{A}_\Sigma$ be the corresponding Hodge star operator. By $\star$ we will also denote the linear automorphism $\star : C^\infty(S^1, \mathcal{A}_\Sigma) \to C^\infty(S^1, \mathcal{A}_\Sigma)$ given by $(\star A^\perp)(t) = \star(A^\perp(t))$ for all $A^\perp \in \mathcal{A}^\perp$ and $t \in S^1$. We then have (cf. Eq. (2.48) in [19])

$$S_{CS}(A^\perp, B) = \pi k \ll A^\perp, \star\big(\tfrac{\partial}{\partial t} + \mathrm{ad}(B)\big) A^\perp \gg_{\mathcal{A}^\perp} + 2\pi k \ll \star A^\perp, dB \gg_{\mathcal{A}^\perp} \tag{2.13}$$

for all $B \in \mathcal{B}$ and $A^\perp \in \mathcal{A}^\perp$, and in particular,

$$S_{CS}(\check{A}^\perp, B) = \pi k \ll \check{A}^\perp, \star\big(\tfrac{\partial}{\partial t} + \mathrm{ad}(B)\big) \check{A}^\perp \gg_{\mathcal{A}^\perp} \tag{2.14}$$

$$S_{CS}(A^\perp_c, B) = 2\pi k \ll \star A^\perp_c, dB \gg_{\mathcal{A}^\perp} \tag{2.15}$$

for $B \in \mathcal{B}$, $\check{A}^\perp \in \check{\mathcal{A}}^\perp$, and $A^\perp_c \in \mathcal{A}^\perp_c$.

2.4 Ribbon version of Eq. (2.7)

Recall that our goal is to find a rigorous realization of Witten's CS path integral expressions which reproduces the Reshetikhin-Turaev invariants (in the special situation described in the Introduction). Since the Reshetikhin-Turaev invariants are defined for ribbon links (or, equivalently[6], for framed links) we will now write down a ribbon analogue of Eq. (2.7).

A closed ribbon R in $\Sigma \times S^1$ is a smooth embedding $R : S^1 \times [0,1] \to \Sigma \times S^1$. A ribbon link in $\Sigma \times S^1$ is a finite tuple of non-intersecting closed ribbons in in $\Sigma \times S^1$. We will replace the link $L = (l_1, l_2, \ldots, l_m)$ by a ribbon link $L_{ribb} = (R_1, R_2, \ldots, R_m)$ where each R_i, $i \leq m$, is chosen such that $l_i(t) = R_i(t, 1/2)$ for all $t \in S^1$. Instead of L_{ribb} we will simply write L in the following. From now on we will assume that $\sigma_0 \in \Sigma$ was chosen such that

$$\sigma_0 \notin \bigcup\nolimits_i \mathrm{Image}(R^i_\Sigma)$$

where $R^i_\Sigma := \pi_\Sigma \circ R_i$. For every $R \in \{R_1, R_2, \ldots, R_m\}$ we define

$$\mathrm{Hol}_R(A) := \lim_{n \to \infty} \prod\nolimits_{j=1}^n \exp\big(\tfrac{1}{n} \int_0^1 A(l'_u(t)) du\big)_{|t = j/n} \in G$$

where l_u, $u \in [0,1]$, is the knot $l_u := R(\cdot, u)$, considered as a loop $[0,1] \to \Sigma \times S^1$. Moreover, for $A^\perp \in \mathcal{A}^\perp$ and $B \in \mathcal{B}$ we set

$$\begin{aligned}\mathrm{Hol}_R(A^\perp, B) &:= \mathrm{Hol}_R(A^\perp + B dt) \\ &= \lim_{n \to \infty} \prod\nolimits_{j=1}^n \exp\big(\tfrac{1}{n} \int_0^1 [A^\perp(l^u_{S^1}(t))((l^u_\Sigma)'(t)) + B(l^u_\Sigma(t)) \cdot dt((l^u_{S^1})'(t))] du\big)_{|t = j/n}\end{aligned} \tag{2.16}$$

[6]From the knot theory point of view the framed link picture and the ribbon link picture are equivalent. However, the ribbon picture seems to be better suited for the study of the Chern-Simons path integral in the torus gauge

where $l_{S^1}^u := \pi_{S^1} \circ l_u$ and $l_\Sigma^u := \pi_\Sigma \circ l_u$ for each $u \in [0,1]$.

We now obtain the aforementioned ribbon analogue of Eq. (2.7) by replacing the expression $\mathrm{Hol}_{l_i}(\check{A}^\perp + A_c^\perp, B)$ in Eq. (2.7) with $\mathrm{Hol}_{R_i}(\check{A}^\perp + A_c^\perp, B)$:

$$\mathrm{WLO}(L) \sim \sum_{y \in I} \int_{\mathcal{A}_c^\perp \times \mathcal{B}} \Big\{ 1_{C^\infty(\Sigma, \mathfrak{t}_{reg})}(B) \,\mathrm{Det}(B) \\ \times \Big[\int_{\check{\mathcal{A}}^\perp} \Big(\prod_i \mathrm{Tr}_{\rho_i}\big(\mathrm{Hol}_{R_i}(\check{A}^\perp + A_c^\perp, B)\big)\Big) \, d\mu_B^\perp(\check{A}^\perp) \Big] \\ \times \exp\big(-2\pi i k \langle y, B(\sigma_0)\rangle\big) \Big\} \exp(iS_{CS}(A_c^\perp, B))(DA_c^\perp \otimes DB) \quad (2.17)$$

where, as a preparation for Sec. 2.5, we have set, for each $B \in \mathcal{B}$,

$$\check{Z}(B) := \int \exp(iS_{CS}(\check{A}^\perp, B)) D\check{A}^\perp, \quad (2.18)$$

$$d\mu_B^\perp := \tfrac{1}{\check{Z}(B)} \exp(iS_{CS}(\check{A}^\perp, B)) D\check{A}^\perp \quad (2.19)$$

and

$$\mathrm{Det}(B) := \mathrm{Det}_{FP}(B) \check{Z}(B) \quad (2.20)$$

2.5 Rewriting $\mathrm{Det}(B)$

Informally, we have for $B \in \mathcal{B}_{reg} := C^\infty(\Sigma, \mathfrak{t}_{reg})$

$$\check{Z}(B) \sim \det\big(\tfrac{\partial}{\partial t} + \mathrm{ad}(B)\big)^{-1/2} \overset{(*)}{\sim} \det\Big(\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B))_{|\mathfrak{k}}\big)^{(1)}\Big)^{-1/2} \quad (2.21)$$

where $\frac{\partial}{\partial t} + \mathrm{ad}(B) : \check{\mathcal{A}}^\perp \to \check{\mathcal{A}}^\perp$ is the operator appearing in Eq. (2.14) above, and where $(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B))_{|\mathfrak{k}})^{(1)}$ is the linear operator on $\mathcal{A}_{\Sigma,\mathfrak{k}} = \Omega^1(\Sigma, \mathfrak{k})$ given by Eq. (2.12) above with $p = 1$. Here step $(*)$ is suggested by

$$\det\big(\tfrac{\partial}{\partial t} + \mathrm{ad}(b)\big) \sim \det\big((1_{\mathfrak{k}} - \exp(\mathrm{ad}(b))_{|\mathfrak{k}})\big) \quad \forall b \in \mathfrak{t}_{reg}$$

where $\frac{\partial}{\partial t} + \mathrm{ad}(b) : C^\infty(S^1, \mathfrak{k}) \to C^\infty(S^1, \mathfrak{k})$ and where $\det(\frac{\partial}{\partial t} + \mathrm{ad}(b))$ is defined with the help of a standard ζ-function regularization argument.

Observe also that $\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B))_{|\mathfrak{k}}\big)^{(0)} = \star^{-1} \circ \big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B))_{|\mathfrak{k}}\big)^{(2)} \circ \star$ where $(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B))_{|\mathfrak{k}})^{(2)}$ is the linear operator on $\Omega^2(\Sigma, \mathfrak{k})$ given by Eq. (2.12) above with $p = 2$ and where $\star : \Omega^0(\Sigma, \mathfrak{k}) \to \Omega^2(\Sigma, \mathfrak{k})$ is the Hodge star operator induced by $\mathbf{g}_\Sigma$. Thus we obtain, informally,

$$\det\Big(\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B))_{|\mathfrak{k}}\big)^{(0)}\Big) = \det\Big(\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B))_{|\mathfrak{k}}\big)^{(2)}\Big) \quad (2.22)$$

Combining Eq. (2.20), Eq. (2.21), and Eq. (2.22) we obtain

$$\mathrm{Det}(B) = \prod_{p=0}^{2} \det\Big(\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B))_{|\mathfrak{k}}\big)^{(p)}\Big)^{(-1)^p/2} \quad (2.23)$$

3 Simplicial realization $\mathrm{WLO}_{rig}^{disc}(L)$ of $\mathrm{WLO}(L)$

3.1 Some polyhedral cell complexes

Let $\mathcal{P}$ be a finite oriented polyhedral cell complex (cf. Appendix C in [19]).

- We denote by $\mathfrak{F}_p(\mathcal{P})$, $p \in \mathbb{N}_0$, the set of p-faces of $\mathcal{P}$. The elements of $\mathfrak{F}_0(\mathcal{P})$ ($\mathfrak{F}_1(\mathcal{P})$, respectively) will be called the "vertices" ("edges", respectively) of $\mathcal{P}$.

- For every fixed real vector space V we denote by $C^p(\mathcal{P}, V)$, $p \in \mathbb{N}_0$, the space of maps $\mathfrak{F}_p(\mathcal{P}) \to V$ ("V-valued p-cochains of $\mathcal{P}$"). Instead of $C^p(\mathcal{P}, \mathbb{R})$ we will often write $C_p(\mathcal{P})$.

- We identify $\mathfrak{F}_p(\mathcal{P})$, $p \in \mathbb{N}_0$, with a subset of $C_p(\mathcal{P}) = C^p(\mathcal{P}, \mathbb{R})$ in the obvious way, i.e. each $\alpha \in \mathfrak{F}_p(\mathcal{P})$ is identified with $\delta_\alpha \in C^p(\mathcal{P}, \mathbb{R})$ given by $\delta_\alpha(\beta) = \delta_{\alpha\beta}$ for all $\beta \in \mathfrak{F}_p(\mathcal{P})$.

- By $d_{\mathcal{P}}$, $p \in \mathbb{N}_0$, we will denote the coboundary operator[7] $C^p(\mathcal{P}, V) \to C^{p+1}(\mathcal{P}, V)$.

i) As a discrete analogue of the Lie group S^1 we will use the finite cyclic group $\mathbb{Z}_N$, $N \in \mathbb{N}$. The number N will be kept fixed throughout the rest of this paper. We will identify $\mathbb{Z}_N$ with the subgroup $\{e^{2\pi i k/N} \mid 1 \leq k \leq N\}$ of S^1. The points of $\mathbb{Z}_N$ induce a polyhedral cell decomposition of S^1. The (1-dimensional oriented[8]) polyhedral cell complex obtained in this way will also be denoted by $\mathbb{Z}_N$ in the following.

ii) We fix a finite oriented smooth polyhedral cell decomposition $\mathcal{C}$ of Σ . By $\mathcal{C}'$ we will denote the "canonical dual" of the polyhedral cell decomposition $\mathcal{C}$ (cf. the end of Appendix C in [19]), again equipped with an orientation. By $\mathcal{K}$ and $\mathcal{K}'$ we will denote the (oriented) polyhedral cell complexes associated to $\mathcal{C}$ and $\mathcal{C}'$, i.e. $\mathcal{K} = (\Sigma, \mathcal{C})$ and $\mathcal{K}' = (\Sigma, \mathcal{C}')$. Instead of $\mathcal{K}$ and $\mathcal{K}'$ we often write K_1 and K_2 and we set $K := (K_1, K_2)$.

iii) We introduce a joint subdivision $q\mathcal{K}$ of $\mathcal{K}$ and $\mathcal{K}'$ which is uniquely determined by the conditions

$$\begin{aligned}
&\mathfrak{F}_0(q\mathcal{K}) = \mathfrak{F}_0(b\mathcal{K}),\\
&\mathfrak{F}_1(q\mathcal{K}) = \mathfrak{F}_1(b\mathcal{K}) \backslash \{e \in \mathfrak{F}_1(b\mathcal{K}) \mid \text{ both endpoints of } e \text{ lie in } \mathfrak{F}_0(\mathcal{K}) \cup \mathfrak{F}_0(\mathcal{K}')\},
\end{aligned}$$

$b\mathcal{K}$ being the barycentric subdivision of $\mathcal{K}$ (cf. Sec. 4.4.3 in [19] for more details). We equip the faces of $q\mathcal{K}$ with an orientation. For convenience we choose the orientation on the edges of $q\mathcal{K}$ to be "compatible"[9] with the orientation on the edges of $\mathcal{K}$ and $\mathcal{K}'$.

iv) By $\mathcal{K} \times \mathbb{Z}_N$ and $q\mathcal{K} \times \mathbb{Z}_N$ we will denote the obvious product (polyhedral) cell complexes.

3.2 The basic spaces

a) The spaces $\mathcal{B}(q\mathcal{K})$, $\mathcal{A}_\Sigma(q\mathcal{K})$, and $\mathcal{A}^\perp(q\mathcal{K})$

We first introduce the following simplicial analogues of the spaces $\mathcal{B}$, $\mathcal{A}_\Sigma$, and $\mathcal{A}^\perp$ in Sec. 2.1 above:

$$\mathcal{B}(qK) := C^0(q\mathcal{K}, \mathfrak{t}) \tag{3.1a}$$
$$\mathcal{A}_\Sigma(q\mathcal{K}) := C^1(q\mathcal{K}, \mathfrak{g}) \tag{3.1b}$$
$$\mathcal{A}^\perp(q\mathcal{K}) := \mathrm{Map}(\mathbb{Z}_N, \mathcal{A}_\Sigma(q\mathcal{K})) \tag{3.1c}$$

The scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ induces scalar products $\ll \cdot, \cdot \gg_{\mathcal{B}(q\mathcal{K})}$ and $\ll \cdot, \cdot \gg_{\mathcal{A}_\Sigma(q\mathcal{K})}$ on $\mathcal{B}(q\mathcal{K})$ and $\mathcal{A}_\Sigma(q\mathcal{K})$ in the standard way. We define a scalar product $\ll \cdot, \cdot \gg_{\mathcal{A}^\perp(q\mathcal{K})}$ on $\mathcal{A}^\perp(q\mathcal{K}) = \mathrm{Map}(\mathbb{Z}_N, \mathcal{A}_\Sigma(q\mathcal{K}))$ by

$$\ll A_1^\perp, A_2^\perp \gg_{\mathcal{A}^\perp(q\mathcal{K})} = \tfrac{1}{N} \sum\nolimits_{t \in \mathbb{Z}_N} \ll A_1^\perp(t), A_2^\perp(t) \gg_{\mathcal{A}_\Sigma(q\mathcal{K})} \tag{3.2}$$

for all $A_1^\perp, A_2^\perp \in \mathcal{A}^\perp(q\mathcal{K})$.

---

[7] in the special case $p = 0$, which is the only case relevant for us, $d_{\mathcal{P}} : C^0(\mathcal{P}, V) \to C^1(\mathcal{P}, V)$ is given explicitly by $df(e) = f(end(e)) - f(start(e))$ for all $f \in C^0(\mathcal{P}, V)$ and $e \in \mathfrak{F}_1(\mathcal{P})$ where $start(e), end(e) \in \mathfrak{F}_0(\mathcal{P})$ denote the starting/end point of the (oriented) edge e

[8] we equip each edge of $\mathbb{Z}_N$ with the orientation induced by the orientation dt of S^1

[9] more precisely, for each $e \in \mathfrak{F}_1(q\mathcal{K})$ we choose the orientation which is induced by orientation of the unique edge $e' \in \mathfrak{F}_1(\mathcal{K}) \cup \mathfrak{F}_1(\mathcal{K}')$ which contains e

b) The subspaces $\mathcal{B}(\mathcal{K})$, $\mathcal{A}_{\Sigma}(K)$, and $\mathcal{A}^{\perp}(K)$

For technical reasons (cf. Remark 3.1 below) we will now introduce suitable subspaces of $\mathcal{B}(q\mathcal{K})$, $\mathcal{A}_{\Sigma}(q\mathcal{K})$, and $\mathcal{A}^{\perp}(q\mathcal{K})$. It will be convenient to first define these three spaces in an "abstract" way and then to explain how they are embedded into the three aforementioned spaces. We set

$$\mathcal{B}(\mathcal{K}) := C^0(\mathcal{K}, \mathfrak{t}) \tag{3.3a}$$

$$\mathcal{A}_{\Sigma}(K) := C^1(K_1, \mathfrak{g}) \oplus C^1(K_2, \mathfrak{g}) \tag{3.3b}$$

$$\mathcal{A}^{\perp}(K) := \mathrm{Map}(\mathbb{Z}_N, \mathcal{A}_{\Sigma}(K)) \tag{3.3c}$$

- We will identify $\mathcal{A}_{\Sigma}(K) \cong (C_1(K_1) \oplus C_1(K_2)) \otimes_{\mathbb{R}} \mathfrak{g}$ with a linear subspace of $\mathcal{A}_{\Sigma}(q\mathcal{K}) \cong C_1(q\mathcal{K}) \otimes_{\mathbb{R}} \mathfrak{g}$ by means of the linear injection $\psi \otimes \mathrm{id}_{\mathfrak{g}}$ where $\psi : C_1(K_1) \oplus C_1(K_2) \to C_1(q\mathcal{K})$ is the linear injection given by $\psi(e) = e_1 + e_2$ for all $e \in \mathfrak{F}_1(K_1) \cup \mathfrak{F}_1(K_2)$ where $e_1, e_2 \in \mathfrak{F}_1(q\mathcal{K})$ are the two edges of $q\mathcal{K}$ "contained" in e.

- Since $\mathcal{A}_{\Sigma}(K)$ is now identified with a subspace of $\mathcal{A}_{\Sigma}(q\mathcal{K})$ the space $\mathcal{A}^{\perp}(K)$ can be considered as a subspace of $\mathcal{A}^{\perp}(q\mathcal{K})$ in the obvious way.

- Finally, the space $\mathcal{B}(\mathcal{K})$ will be identified with a subspace of $\mathcal{B}(q\mathcal{K})$ via the linear injection $\psi : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(q\mathcal{K})$ which associates to each $B \in \mathcal{B}(\mathcal{K})$ the extension $\bar{B} \in \mathcal{B}(q\mathcal{K})$ given by $\bar{B}(x) = \mathrm{mean}_{y \in S_x} B(y)$ for all $x \in \mathfrak{F}_0(q\mathcal{K})$. Here "mean" refers to the arithmetic mean and S_x denotes the set of all $y \in \mathfrak{F}_0(\mathcal{K})$ which lie in the closure of the unique open cell of $\mathcal{K}$ containing x.

Remark 3.1 i) The reason for introducing the subspaces $\mathcal{A}_{\Sigma}(K)$ and $\mathcal{A}^{\perp}(K)$ is that these spaces allow us to obtain a nice simplicial analogue of the Hodge star operator, cf. Sec. 3.4 below.

ii) In order to motivate the introduction of the subspace $\mathcal{B}(\mathcal{K})$ of $\mathcal{B}(q\mathcal{K})$ we remark that $\ker(\pi \circ d_{q\mathcal{K}}) \neq \mathcal{B}_c(q\mathcal{K})$ where $\mathcal{B}_c(q\mathcal{K}) := \{B \in \mathcal{B}(q\mathcal{K}) \mid B \text{ constant}\}$ and where

$$\pi : \mathcal{A}_{\Sigma}(q\mathcal{K}) \to \mathcal{A}_{\Sigma}(K) \tag{3.4}$$

denotes the orthogonal projection. The advantage of working with the space $\mathcal{B}(\mathcal{K})$ is that

$$\ker((\pi \circ d_{q\mathcal{K}})_{|\mathcal{B}(\mathcal{K})}) = \mathcal{B}_c(q\mathcal{K}) \tag{3.5}$$

(Observe that $\mathcal{B}_c(q\mathcal{K}) \subset \mathcal{B}(\mathcal{K})$). Eq. (3.5) will play an important role in Step 2 in the proof of Theorem 5.7 below.

c) The spaces $\check{\mathcal{A}}^{\perp}(K)$ and $\mathcal{A}^{\perp}_c(K)$

In order to obtain a simplicial analogue of the decomposition $\mathcal{A}^{\perp} = \check{\mathcal{A}}^{\perp} \oplus \mathcal{A}^{\perp}_c$ in Eq. (2.3) above we introduce the following spaces:

$$\mathcal{A}_{\Sigma,\mathfrak{t}}(K) := C^1(K_1, \mathfrak{t}) \oplus C^1(K_2, \mathfrak{t}) \tag{3.6a}$$

$$\mathcal{A}_{\Sigma,\mathfrak{k}}(K) := C^1(K_1, \mathfrak{k}) \oplus C^1(K_2, \mathfrak{k}) \tag{3.6b}$$

$$\check{\mathcal{A}}^{\perp}(K) := \{A^{\perp} \in \mathcal{A}^{\perp}(K) \mid \textstyle\sum_{t \in \mathbb{Z}_N} A^{\perp}(t) \in \mathcal{A}_{\Sigma,\mathfrak{k}}(K)\} \tag{3.6c}$$

$$\mathcal{A}^{\perp}_c(K) := \{A^{\perp} \in \mathcal{A}^{\perp}(K) \mid \ A^{\perp}(\cdot) \text{ is constant and } \mathcal{A}_{\Sigma,\mathfrak{t}}(K)\text{-valued}\} \overset{(*)}{\cong} \mathcal{A}_{\Sigma,\mathfrak{t}}(K) \tag{3.6d}$$

where in step $(*)$ we made the obvious identification. Observe that we have

$$\mathcal{A}^{\perp}(K) = \check{\mathcal{A}}^{\perp}(K) \oplus \mathcal{A}^{\perp}_c(K). \tag{3.7}$$

Convention 3.2 In the following we will always consider $\mathcal{B}(\mathcal{K})$, $\mathcal{A}^{\perp}(K)$ and their subspaces as Euclidean vector spaces in the "obvious"[10] way.

3.3 Discrete analogue of the operator $\frac{\partial}{\partial t} + \mathrm{ad}(B)$ in Eq. (2.13)

a) Discrete analogue(s) of the operator $\frac{\partial}{\partial t} + \mathrm{ad}(b) : C^{\infty}(S^1, \mathfrak{g}) \to C^{\infty}(S^1, \mathfrak{g})$, $b \in \mathfrak{t}$

As a preparation for the next subsection, let us introduce, for fixed $b \in \mathfrak{t}$, two simplicial analogues $\hat{L}^{(N)}(b) : \mathrm{Map}(\mathbb{Z}_N, \mathfrak{g}) \to \mathrm{Map}(\mathbb{Z}_N, \mathfrak{g})$ and $\check{L}^{(N)}(b) : \mathrm{Map}(\mathbb{Z}_N, \mathfrak{g}) \to \mathrm{Map}(\mathbb{Z}_N, \mathfrak{g})$ of the continuum operator $L(b) := \frac{\partial}{\partial t} + \mathrm{ad}(b) : C^{\infty}(S^1, \mathfrak{g}) \to C^{\infty}(S^1, \mathfrak{g})$ by

$$\hat{L}^{(N)}(b) := N(\tau_1 e^{\mathrm{ad}(b)/N} - \tau_0) \tag{3.8}$$

$$\check{L}^{(N)}(b) := N(\tau_0 - \tau_{-1} e^{-\mathrm{ad}(b)/N}) \tag{3.9}$$

where τ_x, for $x \in \mathbb{Z}_N$, denotes the translation operator $\mathrm{Map}(\mathbb{Z}_N, \mathfrak{g}) \to \mathrm{Map}(\mathbb{Z}_N, \mathfrak{g})$ given by $(\tau_x f)(t) = f(t + x)$ for all $t \in \mathbb{Z}_N$. I want to emphasize that $\hat{L}^{(N)}(b)$ and $\check{L}^{(N)}(b)$ are indeed totally natural simplicial analogues of $L(b)$, see Sec. 5 in [19] for a detailed motivation.

Remark 3.3 The operator $\bar{L}^{(N)}(b) : \mathrm{Map}(\mathbb{Z}_N, \mathfrak{g}) \to \mathrm{Map}(\mathbb{Z}_N, \mathfrak{g})$ given by

$$\bar{L}^{(N)}(b) := \tfrac{N}{2}(\tau_1 e^{\mathrm{ad}(b)/N} - \tau_{-1} e^{-\mathrm{ad}(b)/N})$$

might at first look appear to be the natural candidate for a simplicial analogue of the continuum operator $L(b)$. However, there are several problems with $\bar{L}^{(N)}(b)$. Firstly, the properties of $\bar{L}^{(N)}(b)$ depend on whether N is odd or even. Secondly, when N is odd then $\bar{L}^{(N)}(b)$ seems to have the "wrong" determinant. Most probably, part ii) of Remark 3.4 below will no longer be true if we redefine the operator $L^{(N)}(B)$ given in Eq. (3.11) below using $\bar{L}^{(N)}(b)$ instead of $\hat{L}^{(N)}(b)$ and $\check{L}^{(N)}(b)$. On the other hand, if N is even then $\bar{L}^{(N)}(b)$ has the "wrong"[11] kernel.

b) Discrete analogue of the operator $\frac{\partial}{\partial t} + \mathrm{ad}(B) : \mathcal{A}^{\perp} \to \mathcal{A}^{\perp}$

For every fixed $B \in \mathcal{B}(q\mathcal{K})$ we first introduce the linear operators

$$\hat{L}^{(N)}(B) \text{ on } \mathrm{Map}(\mathbb{Z}_N, C^1(K_1, \mathfrak{g})) \cong \oplus_{e \in \mathfrak{F}_1(K_1)} \mathrm{Map}(\mathbb{Z}_N, \mathfrak{g}), \text{ and}$$
$$\check{L}^{(N)}(B) \text{ on } \mathrm{Map}(\mathbb{Z}_N, C^1(K_2, \mathfrak{g})) \cong \oplus_{e \in \mathfrak{F}_1(K_2)} \mathrm{Map}(\mathbb{Z}_N, \mathfrak{g})$$

which are given by

$$\hat{L}^{(N)}(B) \cong \oplus_{e \in \mathfrak{F}_1(K_1)} \hat{L}^{(N)}(B(\bar{e})) \tag{3.10a}$$

$$\check{L}^{(N)}(B)) \cong \oplus_{e \in \mathfrak{F}_1(K_2)} \check{L}^{(N)}(B(\bar{e})) \tag{3.10b}$$

where $\bar{e} \in \mathfrak{F}_0(q\mathcal{K})$ for $e \in \mathfrak{F}_1(K_1) \cup \mathfrak{F}_1(K_2)$ is the barycenter of e.

As the simplicial analogue of the the operator $\frac{\partial}{\partial t} + \mathrm{ad}(B) : \mathcal{A}^{\perp} \to \mathcal{A}^{\perp}$ we now take the operator $L^{(N)}(B) : \mathcal{A}^{\perp}(K) \to \mathcal{A}^{\perp}(K)$ which, under the identification

$$\mathcal{A}^{\perp}(K) \cong \mathrm{Map}(\mathbb{Z}_N, C^1(K_1, \mathfrak{g})) \oplus \mathrm{Map}(\mathbb{Z}_N, C^1(K_2, \mathfrak{g})),$$

[10] More precisely, we will assume that the space $\mathcal{B}(\mathcal{K})$ (or any subspace of $\mathcal{B}(\mathcal{K})$) is equipped with the (restriction of the) scalar product $\ll \cdot, \cdot \gg_{\mathcal{B}(q\mathcal{K})}$ on $\mathcal{B}(q\mathcal{K})$, and the space $\mathcal{A}^{\perp}(K)$ (or any subspace of $\mathcal{A}^{\perp}(K)$) is equipped with the restriction of the scalar product $\ll \cdot, \cdot \gg_{\mathcal{A}^{\perp}(q\mathcal{K})}$, introduced in Sec. 3.2 above

[11] For example, if $b \in \mathfrak{t}_{reg}$ then we have $\ker(L(b)) = \{f \in C^{\infty}(S^1, \mathfrak{g}) \mid f \text{ constant and } \mathfrak{t}\text{-valued}\}$. Similarly, we have $\ker(\hat{L}^{(N)}(b)) = \ker(\check{L}^{(N)}(b)) = \{f \in \mathrm{Map}(\mathbb{Z}_N, \mathfrak{g}) \mid f \text{ constant and } \mathfrak{t}\text{-valued}\}$. By contrast, $\ker(\bar{L}^{(N)}(b))$ is strictly larger than $\{f \in \mathrm{Map}(\mathbb{Z}_N, \mathfrak{g}) \mid f \text{ constant and } \mathfrak{t}\text{-valued}\}$.

is given by (cf. Remark 3.4 below for the motivation)

$$L^{(N)}(B) = \begin{pmatrix} \hat{L}^{(N)}(B) & 0 \\ 0 & \check{L}^{(N)}(B) \end{pmatrix} \tag{3.11}$$

Note that $L^{(N)}(B)$ leaves the subspace $\check{\mathcal{A}}^{\perp}(K)$ of $\mathcal{A}^{\perp}(K)$ invariant. The restriction of $L^{(N)}(B)$ to $\check{\mathcal{A}}^{\perp}(K)$ will also be denoted by $L^{(N)}(B)$ in the following.

3.4 Definition of $S^{disc}_{CS}(A^{\perp}, B)$

Recall that $\mathcal{K} = K_1$ and $\mathcal{K}' = K_2$ are dual to each other. As in [1] we can therefore introduce simplicial Hodge star operators $\star_{K_1} : C^1(K_1, \mathbb{R}) \to C^1(K_2, \mathbb{R})$ and $\star_{K_2} : C^1(K_2, \mathbb{R}) \to C^1(K_1, \mathbb{R})$. These are the linear isomorphisms given by

$$\star_{K_j} e = \pm \check{e} \qquad \forall e \in \mathfrak{F}_1(K_j) \tag{3.12}$$

for $j = 1, 2$ where $\check{e} \in \mathfrak{F}_1(K_{3-j})$ is the edge dual to e. The sign $\pm$ above is "+" if the orientation of $\check{e}$ is the one induced by the orientation of e and the orientation of Σ, and it is "$-$" otherwise. The two simplicial Hodge star operators above induce "$\mathfrak{g}$-valued versions" $\star_{K_1} : C^1(K_1, \mathfrak{g}) \to C^1(K_2, \mathfrak{g})$ and $\star_{K_2} : C^1(K_2, \mathfrak{g}) \to C^1(K_1, \mathfrak{g})$ in the obvious way.

Let $\star_K$ be the linear automorphism of $\mathcal{A}_{\Sigma}(K) = C^1(K_1, \mathfrak{g}) \oplus C^1(K_2, \mathfrak{g})$ which is given by

$$\star_K := \begin{pmatrix} 0 & \star_{K_2} \\ \star_{K_1} & 0 \end{pmatrix} \tag{3.13}$$

By $\star_K$ we will also denote the linear automorphism of $\mathcal{A}^{\perp}(K)$ given by

$$(\star_K A^{\perp})(t) = \star_K (A^{\perp}(t)) \qquad \forall A^{\perp} \in \mathcal{A}^{\perp}(K), t \in \mathbb{Z}_N \tag{3.14}$$

As the simplicial analogues of the continuum expression $S_{CS}(A^{\perp}, B)$ in Eq. (2.13) above we use the expression

$$S^{disc}_{CS}(A^{\perp}, B) := \pi k \Big[\ll A^{\perp}, \star_K L^{(N)}(B) A^{\perp} \gg_{\mathcal{A}^{\perp}(q\mathcal{K})} + 2 \ll \star_K A^{\perp}, d_{q\mathcal{K}} B \gg_{\mathcal{A}^{\perp}(q\mathcal{K})} \Big] \tag{3.15a}$$

for $B \in \mathcal{B}(q\mathcal{K})$, $A^{\perp} \in \mathcal{A}^{\perp}(K) \subset \mathcal{A}^{\perp}(q\mathcal{K})$. Observe that this implies

$$S^{disc}_{CS}(\check{A}^{\perp}, B) = \pi k \ll \check{A}^{\perp}, \star_K L^{(N)}(B) \check{A}^{\perp} \gg_{\mathcal{A}^{\perp}(q\mathcal{K})} \tag{3.15b}$$

$$S^{disc}_{CS}(A^{\perp}_c, B) = 2\pi k \ll \star_K A^{\perp}_c, d_{q\mathcal{K}} B \gg_{\mathcal{A}^{\perp}(q\mathcal{K})} \tag{3.15c}$$

for $B \in \mathcal{B}(q\mathcal{K})$, $\check{A}^{\perp} \in \check{\mathcal{A}}^{\perp}(K)$, $A^{\perp}_c \in \mathcal{A}^{\perp}_c(K)$.

Remark 3.4 i) The operator $\star_K L^{(N)}(B) : \mathcal{A}^{\perp}(K) \to \mathcal{A}^{\perp}(K)$ is symmetric w.r.t to the scalar product $\ll \cdot, \cdot \gg_{\mathcal{A}^{\perp}(q\mathcal{K})}$, cf. Proposition 5.3 in [19]. This would not be the case if on the RHS of Eq. (3.11) we had used $\hat{L}^{(N)}(B)$ twice (or $\check{L}^{(N)}(B)$ twice).

ii) According to Proposition 5.1 in [20] we have $\det\big(\star_K L^{(N)}(B)_{|\check{\mathcal{A}}^{\perp}(K)}\big) \neq 0$ for all

$$B \in \mathcal{B}_{reg}(q\mathcal{K}) := \{ B \in \mathcal{B}(q\mathcal{K}) \mid B(x) \in \mathfrak{t}_{reg} \text{ for all } x \in \mathfrak{F}_0(q\mathcal{K}) \} \tag{3.16}$$

where $\star_K L^{(N)}(B)_{|\check{\mathcal{A}}^{\perp}(K)}$ is the restriction of $\star_K L^{(N)}(B)$ to the invariant subspace $\check{\mathcal{A}}^{\perp}(K)$ of $\mathcal{A}^{\perp}(K)$.

3.5 Definition of $\mathrm{Hol}^{disc}_R(A^\perp, B)$

a) Preparation: The simplicial loop case

A "simplicial curve" in a finite oriented polyhedral cell complex $\mathcal{P}$ is a finite sequence $c = (x^{(k)})_{0 \le k \le n}$, $n \in \mathbb{N}$, of vertices in $\mathcal{P}$ such that for every $1 \le k \le n$ the two vertices $x^{(k)}$ and $x^{(k-1)}$ either coincide or are the two endpoints of an edge $e \in \mathfrak{F}_1(\mathcal{P})$. We will call n the "length" of the simplicial curve c. If $x^{(n)} = x^{(0)}$ we will call $c = (x^{(k)})_{0 \le k \le n}$ a "simplicial loop" in $\mathcal{P}$.

Every simplicial curve $c = (x^{(k)})_{0 \le k \le n}$ induces a sequence $(e^{(k)})_{1 \le k \le n}$ of "generalized edges", i.e. elements of $\mathfrak{F}_1(\mathcal{P}) \cup \{0\} \cup (-\mathfrak{F}_1(\mathcal{P})) \subset C_1(\mathcal{P})$ in a natural way. More precisely, we have $e^{(k)} = 0$ if $x^{(k-1)} = x^{(k)}$ and $e^{(k)} = \pm e$ if $x^{(k-1)} \neq x^{(k)}$ where $e \in \mathfrak{F}_1(\mathcal{P})$ is the unique edge connecting the vertices $x^{(k-1)}$ and $x^{(k)}$ and where the sign $\pm$ is $+$ if $x^{(k-1)}$ is the starting point of e and $-$ if it is the endpoint.

Convention 3.5 *For a given simplicial loop $l = (x^{(k)})_{0 \le k \le n}$ we will usually write $\bullet\, l^{(k)}$ instead of $x^{(k-1)}$ and $l^{(k)}$ instead of $e^{(k)}$ (for $1 \le k \le n$) where $(e^{(k)})_{1 \le k \le n}$ is the corresponding sequence of generalized edges.*

Let $l = (x^{(k)})_{0 \le k \le n}$, $n \in \mathbb{N}$, be a simplicial loop in $q\mathcal{K} \times \mathbb{Z}_N$ and let $l_{q\mathcal{K}}$ and $l_{\mathbb{Z}_N}$ be the "projected" simplicial loops in $q\mathcal{K}$ and $\mathbb{Z}_N$. Instead of $l_{q\mathcal{K}}$ and $l_{\mathbb{Z}_N}$ we will usually write l_Σ and l_{S^1}. (Recall that Σ and S^1 are the topological spaces underlying $q\mathcal{K}$ and $\mathbb{Z}_N$.)

For $A^\perp \in \mathcal{A}^\perp(K) \subset \mathcal{A}^\perp(q\mathcal{K})$ and $B \in \mathcal{B}(q\mathcal{K})$ we now define the following simplicial analogue of the expression $\mathrm{Hol}_l(A^\perp, B)$ in Eq. (2.9) (cf. Convention 3.5):

$$\mathrm{Hol}^{disc}_l(A^\perp, B) := \prod\nolimits_{k=1}^{n} \exp\Big(A^\perp(\bullet\, l^{(k)}_{S^1})(l^{(k)}_\Sigma) + B(\bullet\, l^{(k)}_\Sigma) \cdot dt^{(N)}(l^{(k)}_{S^1}) \Big) \tag{3.17}$$

with $dt^{(N)} \in C^1(\mathbb{Z}_N, \mathbb{R})$ given by $dt^{(N)}(e) = \frac{1}{N}$ for all $e \in \mathfrak{F}_1(\mathbb{Z}_N)$ and where we made the identification $C^1(\mathbb{Z}_N, \mathbb{R}) \cong \mathrm{Hom}_\mathbb{R}(C_1(\mathbb{Z}_N), \mathbb{R})$ and $\mathcal{A}_\Sigma(q\mathcal{K}) = C^1(q\mathcal{K}, \mathfrak{g}) \cong \mathrm{Hom}_\mathbb{R}(C_1(q\mathcal{K}), \mathfrak{g})$.

b) The simplicial ribbon case

A "closed simplicial ribbon" in a finite oriented polyhedral cell complex $\mathcal{P}$ is a finite sequence $R = (F_i)_{i \le n}$ of 2-faces of $\mathcal{P}$ such that every F_i is a tetragon and such that $F_i \cap F_j = \emptyset$ unless $i = j$ or $j = i \pm 1$ (mod n). In the latter case F_i and F_j intersect in a (full) edge (cf. Remark 4.3 in Sec. 4.3 in [19] and the paragraph before Remark 4.3 in [19]).

From now on we will consider only the special case where $\mathcal{P} = \mathcal{K} \times \mathbb{Z}_N$. Observe that if $R = (F_k)_{k \le \bar{n}}$, $\bar{n} \in \mathbb{N}$, is a closed simplicial ribbon in $\mathcal{K} \times \mathbb{Z}_N$ then either all the edges $e_{ij} := F_i \cap F_j$, $j = i \pm 1$ (mod $\bar{n}$), are parallel to Σ or they are all parallel to S^1. In the first case we will call R "regular".

In the following let $R = (F_k)_{k \le \bar{n}}$, $\bar{n} \in \mathbb{N}$, be a fixed regular closed simplicial ribbon in $\mathcal{K} \times \mathbb{Z}_N$. Observe that R induces three simplicial loops $l^j = (x^{j(k)})_{0 \le k \le n}$, $j = 0, 1, 2$, (with[12] $n \le 2\bar{n}$) in $q\mathcal{K} \times \mathbb{Z}_N$ in a natural way, l^1 and l^2 being the two boundary loops of R and l^0 being the loop "inside" R. [Here we consider R as a subset of $\Sigma \times S^1$ in the obvious way. Note that the vertices $(x^{j(k)})_{0 \le k \le n}$, $j = 0, 1, 2$, appearing above are just the elements of $R \cap \mathfrak{F}_0(q\mathcal{K} \times \mathbb{Z}_N)$. The "starting" points $x^{j(0)}$, $j = 0, 1, 2$, of the three simplicial loops $l^j = (x^{j(k)})_{0 \le k \le n}$ are the three elements of $e \cap \mathfrak{F}_0(q\mathcal{K} \times \mathbb{Z}_N)$ where $e \in \mathfrak{F}_1(\mathcal{K} \times \mathbb{Z}_N)$ is the edge $e = e_{1\bar{n}} = F_1 \cap F_{\bar{n}}$.]

By l^j_Σ and $l^j_{S^1}$, $j = 0, 1, 2$, we will denote the corresponding "projected" simplicial loops in $q\mathcal{K}$ and $\mathbb{Z}_N$.

[12] The common length n of the three loops is given by $n = 2n_\Sigma + n_{S^1}$ where n_Σ (and n_{S_1}, respectively) is the number of those faces appearing in $R = (F_k)_{k \le \bar{n}}$, which are parallel to Σ (or are parallel to S^1, respectively). Observe that since $\bar{n} = n_\Sigma + n_{S^1}$ we have $\bar{n} \le n \le 2\bar{n}$.

Let $A^{\perp} \in \mathcal{A}^{\perp}(K) \subset \mathcal{A}^{\perp}(q\mathcal{K})$ and $B \in \mathcal{B}(q\mathcal{K})$. As the simplicial analogue of the continuum expression $\mathrm{Hol}_R(A^{\perp}, B)$ in Eq. (2.16) we will take

$$\mathrm{Hol}_R^{disc}(A^{\perp}, B) := \prod_{k=1}^{n} \exp\Big(\sum_{j=0}^{2} w(j) \cdot \big((A^{\perp}(\bullet\, l_{S^1}^{j(k)}))(l_{\Sigma}^{j(k)}) + B(\bullet\, l_{\Sigma}^{j(k)}) \cdot dt^{(N)}(l_{S^1}^{j(k)})\big)\Big) \tag{3.18}$$

where we use again Convention 3.5 and where we have introduced three weight factors

$$w(0) = 1/2, \qquad w(1) = 1/4, \qquad w(2) = 1/4$$

Remark 3.6 Other natural choices would be

$$(w(0), w(1), w(2)) = (1/3, 1/3, 1/3) \qquad \text{or} \qquad (w(0), w(1), w(2)) = (0, 1/2, 1/2)$$

However, these two choices would not even lead to the correct values for $\mathrm{WLO}_{rig}^{disc}(L)$ in the special situation of Sec. 5.

3.6 Definition of $\mathrm{Det}^{disc}(B)$

Let us first try the following ansatz for the discrete analogue $\mathrm{Det}^{disc}(B)$ of the heuristic expression $\mathrm{Det}(B)$ given by Eq. (2.23) above. For every $B \in \mathcal{B}_{reg}(q\mathcal{K})$ we set

$$\mathrm{Det}^{disc}(B) := \prod_{p=0}^{2} \Big(\det\big(\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B))_{|\mathfrak{k}}\big)^{(p)}\big)\Big)^{(-1)^p/2} \tag{3.19}$$

where $\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B))_{|\mathfrak{k}}\big)^{(p)} : C^p(\mathcal{K}, \mathfrak{k}) \to C^p(\mathcal{K}, \mathfrak{k})$ is the linear operator given by

$$\big(\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B))_{|\mathfrak{k}}\big)^{(p)}(\alpha)\big)(X) = \big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B(\sigma_X)))_{|\mathfrak{k}}\big) \cdot \alpha(X) \quad \forall \alpha \in C^p(\mathcal{K}, \mathfrak{k}), X \in \mathfrak{F}_p(\mathcal{K}) \tag{3.20}$$

where $\sigma_X \in \mathfrak{F}_0(q\mathcal{K})$ is the barycenter of X. Observe that we can rewrite Eq. (3.19) in the following way:

$$\mathrm{Det}^{disc}(B) = \prod_{p=0}^{2} \Big(\prod_{F \in \mathfrak{F}_p(\mathcal{K})} \det\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B(\bar{F})))_{|\mathfrak{k}}\big)^{1/2}\Big)^{(-1)^p} \tag{3.21}$$

where $\bar{F}$ is the barycenter of F.

It turns out however, that this ansatz would not lead to the correct values for $\mathrm{WLO}_{rig}^{disc}(L)$ defined below. This is why we will modify our original ansatz. In order to do so we first choose a smooth function $\det^{1/2}\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(\cdot))_{|\mathfrak{k}}\big) : \mathfrak{t} \to \mathbb{R}$ with the property $\forall b \in \mathfrak{t} : \big(\det^{1/2}\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(b))_{|\mathfrak{k}}\big)\big)^2 = \det\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(b))_{|\mathfrak{k}}\big)$. Observe that every such function will necessarily take both positive and negative values. Motivated by the formula

$$\det\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(b))_{|\mathfrak{k}}\big) = \prod_{\alpha \in \mathcal{R}} (1 - e^{2\pi i \langle \alpha, b\rangle}) = \prod_{\alpha \in \mathcal{R}_+} \big(4 \sin^2(\pi \langle \alpha, b \rangle)\big)$$

(with $\mathcal{R}$ and $\mathcal{R}_+$ as in Sec. 5.2 below) we will make the choice

$$\det^{1/2}\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(b))_{|\mathfrak{k}}\big) = \prod_{\alpha \in \mathcal{R}_+} \big(2 \sin(\pi \langle \alpha, b \rangle)\big) \tag{3.22}$$

and then redefine $\mathrm{Det}^{disc}(B)$ for $B \in \mathcal{B}_{reg}(q\mathcal{K})$ by

$$\mathrm{Det}^{disc}(B) := \prod_{p=0}^{2} \Big(\prod_{F \in \mathfrak{F}_p(\mathcal{K})} \det^{1/2}\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B(\bar{F})))_{|\mathfrak{k}}\big)\Big)^{(-1)^p} \tag{3.23}$$

Remark 3.7 In the published versions of [19, 20] there is a notational inaccuracy. When we write $\det\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(b))_{|\mathfrak{k}}\big)^{1/2}$ in [19, 20] we actually mean $\det^{1/2}\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(b))_{|\mathfrak{k}}\big)$ given as in Eq. (3.22) above.

3.7 Discrete version of $1_{C^\infty(\Sigma,\mathfrak{t}_{reg})}(B)$

Let us fix a $s > 0$ which is sufficiently small[13] and choose $1^{(s)}_{\mathfrak{t}_{reg}} \in C^\infty(\mathfrak{t}, \mathbb{R})$ such that

- $0 \le 1^{(s)}_{\mathfrak{t}_{reg}} \le 1$
- $1^{(s)}_{\mathfrak{t}_{reg}} = 0$ on a neighborhood of $\mathfrak{t}_{sing} := \mathfrak{t} \backslash \mathfrak{t}_{reg}$
- $1^{(s)}_{\mathfrak{t}_{reg}} = 1$ outside the s-neighborhood of $\mathfrak{t}_{sing}$
- $1^{(s)}_{\mathfrak{t}_{reg}}$ is invariant under the operation of the affine Weyl group $\mathcal{W}_{\text{aff}}$ on $\mathfrak{t}$ (cf. Sec. 5.2 below).

For fixed $B \in \mathcal{B}(q\mathcal{K})$ we will now take the expression

$$\prod\nolimits_x 1^{(s)}_{\mathfrak{t}_{reg}}(B(x)) := \prod\nolimits_{x \in \mathfrak{F}_0(q\mathcal{K})} 1^{(s)}_{\mathfrak{t}_{reg}}(B(x)) \tag{3.24}$$

as the discrete analogue of $1_{C^\infty(\Sigma,\mathfrak{t}_{reg})}(B)$.

3.8 Oscillatory Gauss-type measures

i) An "oscillatory Gauss-type measure" on Euclidean vector space $(V, \langle \cdot, \cdot \rangle)$ is a complex Borel measure $d\mu$ on V of the form

$$d\mu(x) = \tfrac{1}{Z} e^{-\frac{i}{2}\langle x-m, S(x-m)\rangle} dx \tag{3.25}$$

with $Z \in \mathbb{C}\backslash\{0\}$, $m \in V$, and where S is a symmetric endomorphism of V and dx the normalized[14] Lebesgue measure on V. Note that Z, m and S are uniquely determined by $d\mu$. We will often use the notation m_μ and S_μ in order to refer to m and S.

- We call $d\mu$ "centered"iff $m_\mu = 0$.
- We call $d\mu$ "degenerate" iff S_μ is not invertible

ii) Let $d\mu$ be an oscillatory Gauss-type measure on a Euclidean vector space $(V, \langle \cdot, \cdot \rangle)$. A (Borel) measurable function $f : V \to \mathbb{C}$ will be called improperly integrable w.r.t. $d\mu$ iff[15]

$$\int_\sim f d\mu := \int_\sim f(x) d\mu(x) := \lim_{\epsilon \to 0} (\tfrac{\epsilon}{\pi})^{n/2} \int f(x) e^{-\epsilon |x|^2} d\mu(x) \tag{3.26}$$

exists. Here we have set $n := \dim(\ker(S_\mu))$. Note that if $d\mu$ is non-degenerate we have $n = 0$ so the factor $(\frac{\epsilon}{\pi})^{n/2}$ is then trivial.

- We call $d\mu$ "normalized" iff $\int_\sim 1 d\mu = 1$.

[13] s needs to be smaller than the distance between the two sets $\mathfrak{t}_{sing}$ and $\mathfrak{t}_{reg} \cap \frac{1}{k}\Lambda$ where k is as in Sec. 2 and $\Lambda \subset \mathfrak{t}$ is the weight lattice, cf. Sec. 5.2 below

[14] i.e. unit hyper-cubes have volume 1 w.r.t. dx

[15] Observe that $\int_{\ker(S_\mu)} e^{-\epsilon \|x\|^2} dx = (\frac{\epsilon}{\pi})^{-n/2}$. In particular, the factor $(\frac{\epsilon}{\pi})^{n/2}$ in Eq. (3.26) above ensures that also for degenerate oscillatory Gauss-type measure the improper integrals $\int_\sim 1 \, d\mu$ exists

3.9 Simplicial versions of the two Gauss-type measures in Eq. (2.17)

i) As the simplicial analogue of the heuristic complex measure $d\check{\mu}^{\perp}_B = \frac{1}{\check{Z}(B)} \exp(iS_{CS}(\check{A}^{\perp}, B))D\check{A}^{\perp}$ in Eq. (2.17) we will take the (rigorous) complex measure

$$d\check{\mu}^{\perp,disc}_B(\check{A}^{\perp}) := \tfrac{1}{\check{Z}^{disc}(B)} \exp(iS^{disc}_{CS}(\check{A}^{\perp}, B))D\check{A}^{\perp} \tag{3.27}$$

on $\check{\mathcal{A}}^{\perp}(K)$ where $D\check{A}^{\perp}$ denotes the (normalized) Lebesgue measure on $\check{\mathcal{A}}^{\perp}(K)$ and where we have set $\check{Z}^{disc}(B) := \int_{\sim} \exp(iS^{disc}_{CS}(\check{A}^{\perp}, B))D\check{A}^{\perp}$. Observe that $d\check{\mu}^{\perp,disc}_B(\check{A}^{\perp})$ is not well-defined for all B. However, if $B \in \mathcal{B}_{reg}(q\mathcal{K})$, which is the only case relevant for us (cf. Sec. 3.7 above), Eq. (3.15) and Remark 3.4 above imply that the complex measure in Eq. (3.27) is indeed a well-defined, non-degenerate, centered, normalized oscillatory Gauss type measure on $\check{\mathcal{A}}^{\perp}(K)$.

ii) As the simplicial analogue of the heuristic complex measure $\exp(iS_{CS}(A^{\perp}_c, B))(DA^{\perp}_c \otimes DB)$ in Eq. (2.17) we will take the (rigorous) complex measure on $\mathcal{A}^{\perp}_c(K) \oplus \mathcal{B}(\mathcal{K})$

$$\exp(iS^{disc}_{CS}(A^{\perp}_c, B))(DA^{\perp}_c \otimes DB) \tag{3.28}$$

where $DA^{\perp}_c$ denotes the (normalized) Lebesgue measure on $\mathcal{A}^{\perp}_c(K)$ and DB the (normalized) Lebesgue measure on $\mathcal{B}(\mathcal{K})$.

According to Eq. (3.15) above, the (rigorous) complex measure in Eq. (3.28) is a centered oscillatory Gauss type measure on $\mathcal{A}^{\perp}_c(K) \oplus \mathcal{B}(\mathcal{K})$.

3.10 Definition of $\mathrm{WLO}^{disc}_{rig}(L)$ and $\mathrm{WLO}^{disc}_{norm}(L)$

A finite tuple $L = (R_1, R_2, \ldots, R_m)$, $m \in \mathbb{N}$, of closed simplicial ribbons in $\mathcal{K} \times \mathbb{Z}_N$ which do not intersect each other will be called a "simplicial ribbon link" in $\mathcal{K} \times \mathbb{Z}_N$. For every such simplicial ribbon link $L = (R_1, R_2, \ldots, R_m)$ in $\mathcal{K} \times \mathbb{Z}_N$ equipped with a tuple of "colors" $(\rho_1, \rho_2, \ldots, \rho_m)$, $m \in \mathbb{N}$, we now introduce the following simplicial analogue $\mathrm{WLO}^{disc}_{rig}(L)$ of the heuristic expression $\mathrm{WLO}(L)$ in Eq. (2.17):

$$\begin{aligned}\mathrm{WLO}^{disc}_{rig}(L) := \sum\nolimits_{y \in I} \int_{\sim} \Big(\prod\nolimits_x 1^{(s)}_{\mathfrak{t}_{reg}}(B(x))\Big) \operatorname{Det}^{disc}(B) \\ \times \Big[\int_{\sim} \prod\nolimits_{i=1}^m \operatorname{Tr}_{\rho_i}\big(\operatorname{Hol}^{disc}_{R_i}(\check{A}^{\perp} + A^{\perp}_c, B)\big) d\check{\mu}^{\perp,disc}_B(\check{A}^{\perp})\Big] \\ \times \exp\big(-2\pi i k \langle y, B(\sigma_0)\rangle\big) \exp(iS^{disc}_{CS}(A^{\perp}_c, B))(DA^{\perp}_c \otimes DB)\end{aligned} \tag{3.29}$$

where σ_0 is an arbitrary fixed point of $\mathfrak{F}_0(q\mathcal{K})$ which does not lie in $\bigcup_{i \le m} \operatorname{Image}(\pi_{\Sigma} \circ R_i)$. Here we consider each R_i as a continuous map $[0,1] \times S^1 \to \Sigma \times S^1$ in the obvious way (cf. Remark 4.3 in Sec. 4.3 in [19]).

Apart from considering the simplicial analogue $\mathrm{WLO}^{disc}_{rig}(L)$ of the heuristic expression $\mathrm{WLO}(L)$ in Eq. (2.17) it will also be convenient to introduce a simplicial analogue of the normalized heuristic expression

$$\mathrm{WLO}_{norm}(L) := \frac{\mathrm{WLO}(L)}{\mathrm{WLO}(\emptyset)}$$

where $\emptyset$ is the "empty link" in $M = \Sigma \times S^1$. Accordingly, we will now define

$$\mathrm{WLO}^{disc}_{norm}(L) := \frac{\mathrm{WLO}^{disc}_{rig}(L)}{\mathrm{WLO}^{disc}_{rig}(\emptyset)} \tag{3.30}$$

where L is the colored simplicial ribbon link fixed above and where $\mathrm{WLO}^{disc}_{rig}(\emptyset)$ is defined in the obvious way, i.e. by the expression we get from the RHS of Eq. (3.29) after replacing the product $\prod_{i=1}^m \operatorname{Tr}_{\rho_i}\big(\operatorname{Hol}^{disc}_{R_i}(\check{A}^{\perp} + A^{\perp}_c, B)\big)$ by 1.

We conclude this section with four important remarks. In Remark 3.8 we compare the main aims & results of the present paper with those in [19, 20]. In Remark 3.9 we make some comments regarding the case of general ribbon links L. In Remark 3.10 we describe how the main result of the present paper fits into the bigger picture of the "simplicial program" for Chern-Simons theory (cf. also "Goal 1" of the Introduction). Finally, Remark 3.11 we clarify some points related to "Goal 2" of the Introduction.

Remark 3.8 In [19, 20] a simplicial analogue of $\mathrm{WLO}_{norm}(L)$ which is closely related to the simplicial analogue $\mathrm{WLO}^{disc}_{norm}(L)$ above was evaluated explicitly for a simple type of simplicial ribbon links L, cf. Theorem 6.4 in [19]. An analogous result can be obtained for our $\mathrm{WLO}^{disc}_{norm}(L)$ above[16]. More precisely, it can be shown that for every simplicial ribbon link $L = (R_1, R_2, \ldots, R_m)$ in $\mathcal{K} \times \mathbb{Z}_N$ which fulfills an analogue of Conditions (NCP)' and (NH)' in [19] we have $\mathrm{WLO}^{disc}_{norm}(L) = |L|/|\emptyset|$ where $|\cdot|$ is the shadow invariant on $M = \Sigma \times S^1$ associated to $\mathfrak{g}$ and k as above. This result is interesting because it shows how major "building blocks" of the shadow invariant arise within the torus gauge approach to the CS path integral. However, from a knot theoretic point of view the class of simplicial ribbon links L fulfilling (the analogue of) Condition (NCP)' in [19] is not very interesting. In particular, this class of (simplicial ribbon) links does not include any non-trivial knots.

One of the main aims of the present paper is to show that the torus gauge approach to the CS path integral also allows the treatment of non-trivial knots, namely a large class of torus (ribbon) knots in $S^2 \times S^1$, cf. Definition 5.4 and Theorem 5.7 in Sec. 5 below.

Remark 3.9 The simplicial ribbon knots/links $L = (R_1, R_2, \ldots, R_m)$, $m \in \mathbb{N}$, mentioned in Remark 3.8 above have the special property that the projected ribbons $\pi_\Sigma \circ R_i$, $i \leq m$, in Σ have either no (self-)intersections (= the situation in [19, 20]) or only "longitudinal" self-intersections (= the situation in Definition 5.4, Theorem 5.7 and Theorem 5.8 below). As explained in Sec. 6 in [20], if we want to have a chance of obtaining the correct values for the rigorous version of $\mathrm{WLO}_{norm}(L)$ for general simplicial ribbon links $L = (R_1, R_2, \ldots, R_m)$ (where the projected ribbons $\pi_\Sigma \circ R_i$, $i \leq m$, are allowed to have "transversal" intersections) we will probably have to modify our approach in a suitable way. One way to do so is to make what in Sec. 7 in [20] was called the "transition to the BF-theory setting". Alternatively, one can use a "mixed" approach where some of the simplicial spaces are embedded naturally into suitable continuum spaces[17], cf. [21]. This leads to a greater flexibility and allows us, for example, to work with (continuum) Hodge star operators and continuum ribbons instead of the simplicial Hodge star operators and simplicial ribbons mentioned above.

Remark 3.10 The longterm goal of what in Sec. 1 was called the "simplicial program" for Chern-Simons theory (cf. Sec. 3 in [19] and see also [31]) is to find for every oriented closed 3-manifold M and every colored (ribbon) link L in M a rigorous simplicial realization $\mathrm{WLO}^{disc}(L)$ of the original or gauge-fixed CS path integral for the WLOs associated to L such that $\mathrm{WLO}^{disc}(L)$ coincides with the corresponding Reshetikhin-Turaev invariant $RT(M, L)$.

In the present paper we are much less ambitious. Firstly, we only consider the special case $M = \Sigma \times S^1$ (and from Sec. 5 on we restrict ourselves to the case $\Sigma = S^2$) and secondly, we only deal with a restricted class of simplicial ribbon links L (cf. Theorem 5.7 and Theorem 5.8 below).

[16] or rather for $\mathrm{WLO}^{disc}_{norm}(L)$ after making the modification (M2) described in Sec. 3.11 below

[17] For example, we can exploit the embeddings $C^p(\mathcal{K}, V) \hookrightarrow C^p(b\mathcal{K}, V) \overset{W}{\hookrightarrow} \Omega^p(\Sigma^{(2)}, V)$ for $p = 0, 1, 2$ and $V \in \{\mathfrak{g}, \mathfrak{t}\}$, where W is the Whitney map of the simplicial complex $b\mathcal{K}$ and $\Sigma^{(2)}$ is the complement of the 1-skeleton of $b\mathcal{K}$ in Σ. Observe that $b\mathcal{K}$ induces in a natural way a Riemannian metric on $\Sigma^{(2)}$, which gives rise to a Hodge star operator $\star : \Omega^p(\Sigma^{(2)}, V) \to \Omega^{2-p}(\Sigma^{(2)}, V)$

Remark 3.11 In view of "Goal 2" in Comment 1 of the Introduction note that $\mathrm{WLO}^{disc}_{norm}(L)$ can be interpreted as a (convenient) "lattice regularization" of the heuristic continuum expressions $\mathrm{WLO}_{norm}(L)$ above. Usually, when one works with a lattice regularization in Quantum Gauge Field Theory one has to perform a suitable continuum limit. We can do this here as well[18]. So, instead of working with a fixed $\mathcal{K}$ and $\mathbb{Z}_N$ with fixed $N \in \mathbb{N}$ let us now consider a sequence $(\mathcal{K}^{(n)})_{n\in\mathbb{N}}$ of consecutive refinements of $\mathcal{K}$ and $(\mathbb{Z}_{N^{(n)}})_{n\in\mathbb{N}}$ where $N^{(n)} := n \cdot N$. By doing so we can approximate every "horizontal"[19] ribbon link L in $M = \Sigma \times S^1$ by a suitable sequence $(L^{(n)})_{n\in\mathbb{N}}$ of simplicial ribbon links $L^{(n)}$ in $\mathcal{K}^{(n)} \times \mathbb{Z}_{N^{(n)}}$.

Let us now restrict our attention to horizontal ribbon links L in $M = \Sigma \times S^1$ which are analogous to the simplicial ribbon links appearing in Theorem 5.7 and Theorem 5.8 below and let $(L^{(n)})_{n\in\mathbb{N}}$ be a suitable approximating sequence as above. Then we obtain, informally[20],

$$\mathrm{WLO}_{norm}(L) = \lim_{n\to\infty} \mathrm{WLO}^{disc}_{norm}(L^{(n)}) \tag{3.31}$$

where $\mathrm{WLO}^{disc}_{norm}(L^{(n)})$ is defined in an analogous way as $\mathrm{WLO}^{disc}_{norm}(L)$ above (with $\mathcal{K}^{(n)}$ playing the role of $\mathcal{K}$ and $\mathbb{Z}_{N^{(n)}}$ playing the role of $\mathbb{Z}_N$).

But from (the proof of) Theorem 5.7 and Theorem 5.8 it follows that $\mathrm{WLO}^{disc}_{norm}(L^{(n)})$ does not depend on n (provided that $\mathcal{K}$ was chosen fine enough and N large enough). Accordingly, the $n \to \infty$-limit in Eq. (3.31) is trivial and we simply obtain

$$\mathrm{WLO}_{norm}(L) = \mathrm{WLO}^{disc}_{norm}(L^{(1)})$$

So in order to evaluate the heuristic expression $\mathrm{WLO}_{norm}(L)$ (for the special type of continuum ribbon links L we are considering here) it is enough to compute $\mathrm{WLO}^{disc}_{norm}(L^{(1)})$. And this is exactly what is done in Theorem 5.7 and Theorem 5.8 (with $\mathcal{K}^{(1)}$ replaced by $\mathcal{K}$).

3.11 Modification of the definition of $\mathrm{WLO}^{disc}_{rig}(L)$ and $\mathrm{WLO}^{disc}_{norm}(L)$

As we will see later the definition of $\mathrm{WLO}^{disc}_{rig}(L)$ and of $\mathrm{WLO}^{disc}_{norm}(L)$ above need[21] to be modified slightly if we want to obtain the correct values for $\mathrm{WLO}^{disc}_{norm}(L)$. Without such a modification a "wrong" factor $1_{\mathfrak{t}_{reg}}(B(Z_0))$ will appear at the end of the computations in Step 4 in Sec. 5.4 below. Here are two modifications of the current approach for each of which this extra factor does not appear and one indeed obtains the correct values for $\mathrm{WLO}^{disc}_{norm}(L)$:

(M1) Instead of working with closed simplicial ribbons in $\mathcal{K} \times \mathbb{Z}_N$ we could work with closed simplicial ribbons in $q\mathcal{K} \times \mathbb{Z}_N$. In fact, this is exactly what was done in [19, 20] in the situation studied there. The disadvantage of this kind of modification is that the space $\mathcal{B}(\mathcal{K})$ above needs to be replaced by a less natural space. Moreover, the proof of the analogue of Lemma 5.13 in Sec. 5.4 will become unnaturally complicated.

(M2) We regularize the RHS of Eq. (3.29) in a suitable way. In order to do so we first choose a fixed vector $v \in \mathfrak{t}$ which is not orthogonal to any of the roots $\alpha \in \mathcal{R}$. Then we define

[18] There is, however, a major difference compared to the standard situation in QFT where the continuum limit is usually independent of the lattice regularization. In the case of the Chern-Simons path integral (in the torus gauge) the value of the continuum limit will depend on the lattice regularization. In particular, only a distinguished subclass of lattice regularizations will lead to the correct result, cf. [21] for an interpretation of this phenomenon

[19] i.e. a ribbon link in $M = \Sigma \times S^1$ which, when considered as a framed link instead of a ribbon link, is "horizontally framed" in the sense that the framing vector field is "parallel" to the Σ-component of $M = \Sigma \times S^1$

[20] and under the assumption that the simplicial framework we use in Sec. 3 indeed belongs to the "distinguished subclass" of lattice regularizations mentioned in Footnote 18 above

[21] I consider this to be a purely technical issue which can probably be resolved by using an alternative way for making rigorous sense of the RHS of Eq. (2.17), cf. Remark 3.12 below

$B_{displace} \in \mathcal{B}(q\mathcal{K})$ by

$$B_{displace}(x) = \begin{cases} 0 & \text{if } x \in \mathfrak{F}_0(\mathcal{K}) \\ v & \text{if } x \in \mathfrak{F}_0(q\mathcal{K}) \backslash \mathfrak{F}_0(\mathcal{K}) \end{cases}$$

and set, for each $\beta > 0$ and each $B \in \mathcal{B}(\mathcal{K}) \subset \mathcal{B}(q\mathcal{K})$,

$$B(\beta) = B + \beta B_{displace}$$

After that we replace B by $B(\beta)$ in each of the three terms $\prod_x 1^{(s)}_{\mathfrak{t}_{reg}}(B(x)))$, $\mathrm{Det}^{disc}(B)$, and $d\check{\mu}^{\perp,disc}_B(\check{A}^\perp)$ appearing on the RHS of Eq. (3.29). Finally, we let[22] $s \to 0$ and later $\beta \to 0$. More precisely, we add $\lim_{\beta \searrow 0} \lim_{s \searrow 0} \cdots$ in front of the (modified) RHS of Eq. (3.29). ($\mathrm{WLO}^{disc}_{norm}(L)$ is again defined by Eq. (3.30).)

During the proof of Theorem 5.7 below, which will be given in Sec. 5.4 below we will first work with the original definition of $\mathrm{WLO}^{disc}_{rig}(L)$ in Sec. 3.10 until the end of "Step 4". This is instructive because we see how the factor $1_{\mathfrak{t}_{reg}}(B(Z_0))$ arises. After that we switch to the modified definition of $\mathrm{WLO}^{disc}_{rig}(L)$ (using either of the two options (M1) and (M2) above) and complete the proof.

Remark 3.12 The simplicial approach described above for obtaining a rigorous realization of $\mathrm{WLO}(L)$ is simple and fairly natural and it will be sufficient for the goals of the present paper, cf. "Goal 1" and "Goal 2" in Comment 1 in the Introduction.

That said I want to emphasize that even though the approach above is probably one of the simplest ways for making rigorous sense of the RHS of Eq. (2.17) (for the special simplicial ribbon links we are interested in in the present paper) I do not claim that it is the best way of obtaining such a rigorous realization. It is likely that several improvements are possible and that, in particular, there is an alternative to modification (M1) or modification (M2) which is more natural.

4 Some useful results on oscillatory Gauss-type measures

In the present section we will review (without proof and in a slightly modified form) some of the basic definitions and results in [20] on oscillatory Gauss-type measures.

In the following let $(V, \langle \cdot, \cdot \rangle)$ be a Euclidean vector space and $d\mu$ an oscillatory Gauss-type measure on $(V, \langle \cdot, \cdot \rangle)$, cf. Sec. 3.8 above.

Proposition 4.1 *If $d\mu$ is normalized and non-degenerate then we have for all $v, w \in V$*

$$\int_\sim \langle v, x \rangle \ d\mu(x) = \langle v, m \rangle, \qquad \int_\sim \langle v, x \rangle \langle w, x \rangle \ d\mu(x) = \tfrac{1}{i} \langle v, S^{-1} w \rangle + \langle v, m \rangle \langle w, m \rangle \tag{4.1}$$

where $m = m_\mu$ and $S = S_\mu$.

Definition 4.2 *By $\mathcal{P}_{exp}(V)$ we denote the subalgebra of $\mathrm{Map}(V, \mathbb{C})$ which is generated by the polynomial functions $f : V \to \mathbb{C}$ and all functions $f : V \to \mathbb{C}$ of the form $f = \theta \circ \exp_{\mathrm{End}(\mathbb{C}^n)} \circ \varphi$, $n \in \mathbb{N}$, where $\theta : \mathrm{End}(\mathbb{C}^n) \to \mathbb{C}$ is linear, $\varphi : V \to \mathrm{End}(\mathbb{C}^n)$ is affine, and $\exp_{\mathrm{End}(\mathbb{C}^n)} : \mathrm{End}(\mathbb{C}^n) \to \mathrm{End}(\mathbb{C}^n)$ is the exponential map of the (normed) $\mathbb{R}$-algebra $\mathrm{End}(\mathbb{C}^n)$.*

Proposition 4.3 *For every $f \in \mathcal{P}_{exp}(V)$ the improper integral $\int_\sim f \ d\mu \in \mathbb{C}$ exists.*

[22] without letting $s \to 0$ first, the $\beta \to 0$ limit has no effect

Proposition 4.4 *If $d\mu$ is normalized and non-degenerate and if $(Y_k)_{k\le n}$, $n \in \mathbb{N}$, is a sequence of affine maps $V \to \mathbb{R}$ such that*

$$\int_{\sim} Y_i Y_j d\mu = \Big(\int_{\sim} Y_i d\mu\Big)\Big(\int_{\sim} Y_j d\mu\Big) \quad \forall i, j \le n \tag{4.2}$$

then we have for every $\Phi \in \mathcal{P}_{exp}(\mathbb{R}^n)$

$$\int_{\sim} \Phi((Y_k)_k) d\mu = \Phi\Big(\Big(\int_{\sim} Y_k d\mu\Big)_k\Big) \tag{4.3}$$

A totally analogous statement holds in the situation where instead of the sequence $(Y_k)_{k\le n}$, $n \in \mathbb{N}$, we have a family $(Y^a_k)_{k\le n, a\le d}$ of affine maps fulfilling the obvious analogue of Eq. (4.2) and where $\Phi \in \mathcal{P}_{exp}(\mathbb{R}^{n\times d})$.

Definition 4.5 Let $f : V \to \mathbb{C}$ be a continuous function, let $d := \dim(V)$, and let dx be the normalized Lebesgue measure on V. We set

$$\int_V^{\sim} f(x)dx := \tfrac{1}{\pi^{d/2}} \lim_{\epsilon\to 0} \epsilon^{d/2} \int_V e^{-\epsilon|x|^2} f(x)dx \tag{4.4}$$

whenever the expression on the RHS of the previous equation is well-defined.

Remark 4.6 Let Γ be a lattice in V and $f : V \to \mathbb{C}$ a Γ-periodic continuous function. Then $\int_V^{\sim} f(x)dx$ exists and we have

$$\int_V^{\sim} f(x)dx = \frac{1}{vol(Q)} \int_Q f(x)dx \tag{4.5}$$

with $Q := \{\sum_i x_i e_i \mid 0 \le x_i \le 1 \forall i \le d\}$ where $(e_i)_{i\le d}$ is an arbitrary basis of the lattice Γ and where $vol(Q)$ denotes the volume of Q. Clearly, Eq. (4.5) implies

$$\forall y \in V: \quad \int_V^{\sim} f(x)dx = \int_V^{\sim} f(x+y)dx \tag{4.6}$$

Proposition 4.7 *Assume that $V = V_0 \oplus V_1 \oplus V_2$ where V_0, V_1, V_2 are pairwise orthogonal subspaces of V. (We will denote the V_j-component of $x \in V$ by x_j in the following.) Assume also that $d\mu$ is a (centered) normalized oscillatory Gauss-type measure on $(V, \langle\cdot,\cdot\rangle)$ of the form $d\mu(x) = \frac{1}{Z}\exp(i\langle x_2, M x_1\rangle dx$ for some linear isomorphism $M : V_1 \to V_2$. Then, for every $v \in V_2$ and every bounded uniformly continuous function $F : V_0 \oplus V_1 \to \mathbb{C}$ the LHS of the following equation exists iff the RHS exists and in this case we have*

$$\int_{\sim} F(x_0 + x_1)\exp(i\langle x_2, v\rangle) d\mu(x) = \int_{V_0}^{\sim} F(x_0 - M^{-1}v)dx_0, \tag{4.7}$$

where dx_0 is the normalized Lebesgue measure on V_0.

5 Evaluation of $\mathrm{WLO}^{disc}_{rig}(L)$ for torus ribbon knots L in $S^2 \times S^1$

From now on we will only consider the special case $\Sigma = S^2$.

5.1 A certain class of torus (ribbon) knots in $S^2 \times S^1$

Recall that a torus knot in S^3 is a knot which is contained in an unknotted torus $\tilde{\mathcal{T}} \subset S^3$. Motivated by this definition we will now introduce an analogous notion for knots in the manifold $M = S^2 \times S^1$.

Definition 5.1 *A torus knot in $S^2 \times S^1$ of standard type is a knot in $S^2 \times S^1$ which is contained in a torus $\mathcal{T}$ in $S^2 \times S^1$ fulfilling the following condition*

(T) *$\mathcal{T}$ is of the form $\mathcal{T} = \psi(\mathcal{T}_0)$ with $\mathcal{T}_0 := C_0 \times S^1$ where C_0 is an embedded circle in S^2 and $\psi : S^2 \times S^1 \to S^2 \times S^1$ is a diffeomorphism.*

Remark 5.2 Note that every unknotted torus $\tilde{\mathcal{T}}$ in S^3 can be obtained from a torus $\mathcal{T}$ in $S^2 \times S^1$ fulfilling condition (T) by performing a suitable Dehn surgery on a separate knot in $S^2 \times S^1$. Consequently, every torus knot $\tilde{K}$ in S^3 can be obtained from a torus knot K in $S^2 \times S^1$ of standard type by performing such a Dehn surgery. Moreover, even if we restrict ourselves to the special situation where K lies in $\mathcal{T}_0 = C_0 \times S^1$ for C_0 as above we can still obtain all torus knots in S^3 up to equivalence by performing a suitable Dehn surgery. We will exploit this fact in Sec. 6.2 below.

Let us now go back to the simplicial setting introduced in Sec. 3. Recall that in Sec. 3 we fixed two polyhedral cell complexes $\mathcal{K}$ and $\mathbb{Z}_N$ and considered also their product $\mathcal{K} \times \mathbb{Z}_N$. The topological space underlying $\mathcal{K} \times \mathbb{Z}_N$ is $\Sigma \times S^1 = S^2 \times S^1$. We want to find a "simplicial analogue" of Definition 5.1 above. In view of Remark 5.2 we will work with the following definition:

Definition 5.3 *Let l be a simplicial loop in $\mathcal{K} \times \mathbb{Z}_N$ (which we will consider as a continuous map $S^1 \to S^2 \times S^1$ in the obvious way). We say that l is a simplicial torus knot of standard type iff $l : S^1 \to S^2 \times S^1$ is an embedding and also the following condition is fulfilled:*

(TK) Image(l) *is contained in $\mathcal{T}_0 := C_0 \times S^1$ where C_0 is some embedded circle in S^2 which lies on the 1-skeleton of $\mathcal{K}$.*

By $\mathbf{p}(l)$ and $\mathbf{q}(l)$ we will denote the winding numbers of $\pi_i \circ l : S^1 \to S^1$, $i = 1, 2$, where π_1 and π_2 are the two canonical projections $\mathcal{T}_0 = C_0 \times S^1 \cong S^1 \times S^1 \to S^1$ where for the identification $C_0 \cong S^1$ we picked an orientation on C_0. (Observe that $\mathbf{p}(l)$ and $\mathbf{q}(l)$ will always be coprime.)

Definition 5.4 *Let R be a closed simplicial ribbon in $\mathcal{K} \times \mathbb{Z}_N$ (which we will consider as a continuous map $S^1 \times [0, 1] \to S^2 \times S^1$ in the obvious way). We say that R is a simplicial torus ribbon knot of standard type iff it is regular (cf. Sec. 3.5 above) and also the following condition is fulfilled:*

(TRK) *Each of the two simplicial loops l_1 and l_2 on the boundary of R fulfills condition (TK) above.*

The two integers $\mathbf{p} := \mathbf{p}(l_1) = \mathbf{p}(l_2)$ and $\mathbf{q} := \mathbf{q}(l_1) = \mathbf{q}(l_2)$ will be called the winding numbers of R.

Definition 5.5 *Let $R = (F_i)_{i \leq n}$ be a closed simplicial ribbon in $\mathcal{K} \times \mathbb{Z}_N$. We say that R is vertical iff R is regular and moreover, every 2-face $F_i \in \mathfrak{F}_2(\mathcal{K} \times \mathbb{Z}_N)$ is "parallel" to S^1. In this case the three simplicial loops l^j ($j = 0, 1, 2$) in $q\mathcal{K} \times \mathbb{Z}_N$, associated to R (cf. Sec. 3.5 above) will be "parallel" to S^1 as well. More precisely, for each l^j the image of the projected simplicial loop l^j_Σ in $q\mathcal{K}$ will simply consist of a single point $\sigma^j \in \mathfrak{F}_0(q\mathcal{K})$.*

Observe that every vertical closed simplicial ribbon is a simplicial torus ribbon knot of standard type with $\mathbf{p} = 0$ and $\mathbf{q} = \pm 1$. If $\mathbf{q} = 1$ we say that R has standard orientation.

5.2 Some notation

Recall that in Sec. 2, above we have fixed a scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{t}$. Using this scalar product we will now make the obvious identification $\mathfrak{t} \cong \mathfrak{t}^*$.

- $\mathcal{R} \subset \mathfrak{t}^*$ will denote the set of real roots associated to $(\mathfrak{g}, \mathfrak{t})$
- $\check{\mathcal{R}}$ denotes the set of real coroots, i.e. $\check{\mathcal{R}} := \{\check{\alpha} \mid \alpha \in \mathcal{R}\} \subset \mathfrak{t}$ where $\check{\alpha} := \frac{2\alpha}{\langle \alpha, \alpha \rangle}$.
- $\Lambda \subset \mathfrak{t}^*$ denotes the real weight lattice associated to $(\mathfrak{g}, \mathfrak{t})$.
- $\Gamma \subset \mathfrak{t}$ will denote the lattice generated by the set of real coroots.
- A Weyl alcove is a connected component of the set

  $\mathfrak{t}_{reg} = \exp^{-1}(T_{reg}) = \mathfrak{t} \backslash \bigcup_{\alpha \in \mathcal{R}, k \in \mathbb{Z}} H_{\alpha,k}$ where $H_{\alpha,k} := \alpha^{-1}(k)$.
- $\mathcal{W}$ will denote the Weyl group associated to $(\mathfrak{g}, \mathfrak{t})$
- $\mathcal{W}_{\text{aff}}$ will denote the affine Weyl group associated to $(\mathfrak{g}, \mathfrak{t})$, i.e. the group of isometries of $\mathfrak{t} \cong \mathfrak{t}^*$ generated by the orthogonal reflections on the hyperplanes $H_{\alpha,k}$, $\alpha \in \mathcal{R}$, $k \in \mathbb{Z}$, defined above. Equivalently, one can define $\mathcal{W}_{\text{aff}}$ as the group of isometries of $\mathfrak{t} \cong \mathfrak{t}^*$ generated by $\mathcal{W}$ and the translations associated to the coroot lattice Γ. For $\tau \in \mathcal{W}_{\text{aff}}$ we will denote the sign of τ by $(-1)^\tau$.

Recall that in Sec. 2 above we fixed $k \in \mathbb{N}$. Let us now also fix a Weyl chamber $\mathcal{C}$.

- $\mathcal{R}_+$ denotes the set of positive (real) roots associated to $(\mathfrak{g}, \mathfrak{t})$ and $\mathcal{C}$.
- Λ_+ denotes the set of dominant (real) weights associated to $(\mathfrak{g}, \mathfrak{t})$ and $\mathcal{C}$.
- ρ denotes the half-sum of the positive (real) roots
- θ denotes the unique long (real) root in $\overline{\mathcal{C}}$.
- We set $c_{\mathfrak{g}} := 1 + \langle \theta, \rho \rangle$ ($c_{\mathfrak{g}}$ is the dual Coxeter number of $\mathfrak{g}$)
- For $\lambda \in \Lambda_+$ let $\lambda^* \in \Lambda_+$ denote the weight conjugated to λ and $\bar{\lambda} \in \Lambda_+$ the weight conjugated to λ "after applying a shift by ρ". More precisely, $\bar{\lambda}$ is given by $\bar{\lambda} + \rho = (\lambda + \rho)^*$.
- We set $\Lambda_+^k := \{\lambda \in \Lambda_+ \mid \langle \lambda + \rho, \theta \rangle < k\} = \{\lambda \in \Lambda_+ \mid \langle \lambda, \theta \rangle \leq k - c_{\mathfrak{g}}\}$.

Remark 5.6 In Sec. 1 I mentioned that for a given oriented closed 3-manifold M the Reshetikhin-Turaev invariants associated to $(M, \mathfrak{g}_{\mathbb{C}}, q)$ are widely believed to be equivalent to Witten's heuristic path integral expressions based on the Chern-Simons action function associated to (M, G, k) where G is the simply connected, compact Lie group corresponding to the compact real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$ and $k \in \mathbb{N}$ is chosen suitably. It it commonly believed that this relationship between q and k is given by

$$q = e^{2\pi i/(k + c_{\mathfrak{g}})}, \qquad k \in \mathbb{N}$$

The appearance of $k + c_{\mathfrak{g}}$ instead of k (i.e. the replacement $k \to k + c_{\mathfrak{g}}$) is the famous "shift of the level" k. However, several authors have argued (cf., e.g., [15]) that the occurrence (and magnitude) of such a shift in the level depends on the regularization procedure and renormalization prescription which is used for making sense of the heuristic path integral. Accordingly, it should not be surprising that there are several papers (cf. the references in [15]) where the

shift $k \to k + c_{\mathfrak{g}}$ is not observed and one is therefore led to the following relationship between q and[23] k:

$$q = e^{2\pi i/k}, \qquad k \in \mathbb{N} \text{ with } k > c_{\mathfrak{g}}$$

This is also the case in [19, 20] and the present paper[24].

Let C and S be the $\Lambda_+^k \times \Lambda_+^k$ matrices with complex entries given by

$$C_{\lambda\mu} := \delta_{\lambda\bar{\mu}}, \tag{5.1a}$$

$$S_{\lambda\mu} := \frac{i^{\#\mathcal{R}_+}}{k^{\dim(\mathfrak{t})/2}} \frac{1}{|\Lambda/\Gamma|^{1/2}} \sum\nolimits_{\tau \in \mathcal{W}} (-1)^{\tau} e^{-\frac{2\pi i}{k}\langle \lambda+\rho, \tau\cdot(\mu+\rho)\rangle} \tag{5.1b}$$

for all $\lambda, \mu \in \Lambda_+^k$ where $\#\mathcal{R}_+$ is the number of elements of $\mathcal{R}_+$. We have

$$S^2 = C \tag{5.2}$$

It will be convenient to generalize the definition of $S_{\lambda\mu}$ to the situation of general $\lambda, \mu \in \Lambda$ using again Eq. (5.1b).

Let θ_λ and d_λ for $\lambda \in \Lambda$ be given by[25]

$$\theta_\lambda := e^{\frac{\pi i}{k}\langle \lambda, \lambda+2\rho\rangle} \tag{5.3a}$$

$$d_\lambda := \frac{S_{\lambda 0}}{S_{00}} \overset{(*)}{=} \prod\nolimits_{\alpha \in \mathcal{R}_+} \frac{\sin(\frac{\pi}{k}\langle \lambda + \rho, \alpha\rangle)}{\sin(\frac{\pi}{k}\langle \rho, \alpha\rangle)} \tag{5.3b}$$

where step $(*)$ follows, e.g., from part iii) in Theorem 1.7 in Chap. VI in [10].

For every $\lambda \in \Lambda_+$ we denote by ρ_λ the (up to equivalence) unique irreducible, finite-dimensional, complex representation of G with highest weight λ. For every $\mu \in \Lambda$ we will denote by $m_\lambda(\mu)$ the multiplicity of μ as a weight in ρ_λ. It will be convenient to introduce $\bar{m}_\lambda : \mathfrak{t} \to \mathbb{Z}$ by

$$\bar{m}_\lambda(b) = \begin{cases} m_\lambda(b) & \text{if } b \in \Lambda \\ 0 & \text{otherwise} \end{cases} \tag{5.4}$$

Instead of $\bar{m}_\lambda$ we will simply write m_λ in the following.

Finally, let us define $* : \mathcal{W}_{\text{aff}} \times \mathfrak{t} \to \mathfrak{t}$ by

$$\tau * b = k\big(\tau \cdot \tfrac{1}{k}(b + \rho)\big) - \rho, \qquad \text{for all } \tau \in \mathcal{W}_{\text{aff}} \text{ and } b \in \mathfrak{t} \tag{5.5}$$

and set, for all $\lambda \in \Lambda_+$, $\mu, \nu \in \Lambda$, $\mathbf{p} \in \mathbb{Z}\backslash\{0\}$, and $\tau \in \mathcal{W}_{\text{aff}}$

$$m^{\mu\nu}_{\lambda,\mathbf{p}}(\tau) := (-1)^{\tau} m_\lambda\big(\tfrac{1}{\mathbf{p}}(\mu - \tau * \nu)\big) \in \mathbb{Z} \tag{5.6}$$

and

$$M^{\mu\nu}_{\lambda,\mathbf{p}} := \sum\nolimits_{\tau \in \mathcal{W}_{\text{aff}}} m^{\mu\nu}_{\lambda,\mathbf{p}}(\tau) \in \mathbb{Z} \tag{5.7}$$

[23] In view of the definition of the set Λ_+^k above it is clear that the situation $k \leq c_{\mathfrak{g}}$ is not interesting

[24] by contrast, in [11] a shift $k \to k + c_{\mathfrak{g}}$ was inserted by hand into several formulas. Accordingly, several definitions in the present paper differ from the definitions in [11]

[25] For $r \in \mathbb{Q}$ we will write θ_λ^r instead of $e^{r\cdot\frac{\pi i}{k}\langle \lambda, \lambda+2\rho\rangle}$. Note that this notation is somewhat dangerous since $\theta_{\lambda_1} = \theta_{\lambda_2}$ does, of course, in general not imply $\theta^r_{\lambda_1} = \theta^r_{\lambda_2}$

5.3 The two main results

From now on we will always assume that $k > c_{\mathfrak{g}}$, cf. Remark 5.6 above.

Theorem 5.7 *Let $L = (R_1)$ be a simplicial ribbon link in $\mathcal{K} \times \mathbb{Z}_N$ colored with ρ_1 where R_1 is simplicial torus ribbon knot of standard type with winding numbers $\mathbf{p} \in \mathbb{Z}\backslash\{0\}$ and $\mathbf{q} \in \mathbb{Z}$ (cf. (TRK) in Sec. 5.1). Assume that $\lambda_1 \in \Lambda_+^k$ where λ_1 is the highest weight of ρ_1. Then $\mathrm{WLO}^{disc}_{norm}(L)$ is well-defined and we have*

$$\mathrm{WLO}^{disc}_{norm}(L) = S_{00}^2 \sum\nolimits_{\eta_1,\eta_2 \in \Lambda_+^k} \sum\nolimits_{\tau \in \mathcal{W}_{\mathrm{aff}}} m^{\eta_1\eta_2}_{\lambda_1,\mathbf{p}}(\tau)\ d_{\eta_1} d_{\eta_2}\ \theta_{\eta_1}^{\frac{\mathbf{q}}{\mathbf{p}}} \theta_{\tau*\eta_2}^{-\frac{\mathbf{q}}{\mathbf{p}}} \tag{5.8}$$

The following generalization of Theorem 5.7 will play a crucial role in Sec. 6.2 below.

Theorem 5.8 *Let $L = (R_1, R_2)$ be simplicial ribbon link in $\mathcal{K} \times \mathbb{Z}_N$ colored with (ρ_1, ρ_2) where R_1 is a simplicial torus ribbon knot of standard type with winding numbers $\mathbf{p} \in \mathbb{Z}\backslash\{0\}$ and $\mathbf{q} \in \mathbb{Z}$ and R_2 is vertical with standard orientation. Let us assume that R_1 winds around R_2 in the "positive direction"[26] and that $\lambda_1, \lambda_2 \in \Lambda_+^k$ where λ_1 and λ_2 are the highest weights of ρ_1 and ρ_2. Then $\mathrm{WLO}^{disc}_{norm}(L)$ is well-defined and we have*

$$\mathrm{WLO}^{disc}_{norm}(L) = S_{00} \sum\nolimits_{\eta_1,\eta_2 \in \Lambda_+^k} \sum\nolimits_{\tau \in \mathcal{W}_{\mathrm{aff}}} m^{\eta_1\eta_2}_{\lambda_1,\mathbf{p}}(\tau)\ d_{\eta_1} S_{\lambda_2\eta_2}\ \theta_{\eta_1}^{\frac{\mathbf{q}}{\mathbf{p}}} \theta_{\tau*\eta_2}^{-\frac{\mathbf{q}}{\mathbf{p}}} \tag{5.9}$$

Remark 5.9 In the special case where $\mathbf{p} = 1$ we have $\theta_{\tau*\eta_2}^{-\frac{\mathbf{q}}{\mathbf{p}}} = \theta_{\tau*\eta_2}^{-\mathbf{q}} = \theta_{\eta_2}^{-\mathbf{q}}$ (cf. Eq. (5.64a) below) and Eq. (5.8) can be rewritten as

$$\mathrm{WLO}^{disc}_{norm}(L) = S_{00}^2 \sum\nolimits_{\eta_1,\eta_2 \in \Lambda_+^k} M^{\eta_1\eta_2}_{\lambda_1,1} d_{\eta_1} d_{\eta_2}\ \theta_{\eta_1}^{\mathbf{q}} \theta_{\eta_2}^{-\mathbf{q}}$$

where $M^{\mu\nu}_{\lambda,1}$ is as in Eq. (5.7) above. (A totally analogous remark applies to Eq. (5.9)). But

$$M^{\mu\nu}_{\lambda,1} = \sum\nolimits_{\tau \in \mathcal{W}_{\mathrm{aff}}} (-1)^\tau m_\lambda\big(\mu - \tau * \nu\big) \overset{(*)}{=} N^{\mu}_{\lambda\nu} \tag{5.10}$$

where $N^{\mu}_{\lambda\nu}$, $\lambda, \mu, \nu \in \Lambda_+^k$, are the so-called fusion coefficients, see e.g., [38] for the definition of $N^{\mu}_{\lambda\nu}$ and for the proof of the equality $(*)$ (called the "quantum Racah formula" in [38]).

Eq. (5.10) implies that the RHS of both theorems can be rewritten in terms of Turaev's shadow invariant (or, equivalently, the Reshetikhin-Turaev invariant), cf. Remark 3.8 above. Accordingly, in this case it is clear[27] that both theorems give the results expected in the literature, i.e. Conjecture 1 below is indeed true if $\mathbf{p} = 1$.

5.4 Proof of Theorem 5.7

Let $L = (R_1)$ where R_1 is a simplicial torus ribbon knot of standard type in $\mathcal{K} \times \mathbb{Z}_N$, colored with ρ_1 and with winding numbers $\mathbf{p} \in \mathbb{Z}\backslash\{0\}$ and $\mathbf{q} \in \mathbb{Z}$. (In the following we sometimes write R instead of R_1.) Let $n \in \mathbb{N}$ be the length of the three simplicial loops l^j, $j = 0, 1, 2$, in $q\mathcal{K} \times \mathbb{Z}_N$ associated to $R = R_1$, cf. Sec. 3.5 above.

The symbol $\sim$ will denote equality up to a multiplicative (non-zero) constant which is allowed to depend on G, k, $\mathcal{K}$ and N but not on the colored ribbon knot considered[28].

Recall that, as mentioned in Sec. 3.11 above, until the end of "Step 4" below we will work with the original definition of $\mathrm{WLO}^{disc}_{rig}(L)$. Then we will explain how Steps 1–4 need to be modified if the new definition is used. In Step 5–6 we then work with the new definition of $\mathrm{WLO}^{disc}_{rig}(L)$.

[26] cf. Sec. 5.5 below for a precise definition

[27] Note, for example, that if $\mathbf{p} = 1$ then the simplicial ribbon link $L = (R_1)$ appearing in Theorem 5.7 will fulfill the analogue of Conditions (NCP)' and (NH)' in [19] mentioned in Remark 3.8 above.

[28] in particular, it will depend neither on $\mathbf{p}$ nor on $\mathbf{q}$ nor on ρ_1

a) Step 1: Performing the $\int_\sim \cdots d\check{\mu}_B^{\perp,disc}(\check{A}^\perp)$ integration in Eq. (3.29)

We will prove below that under the assumptions on $L = (R_1)$ made above we have for every fixed $A_c^\perp \in \mathcal{A}_c^\perp(K)$ and $B \in \mathcal{B}_{reg}(q\mathcal{K})$

$$\int_\sim \mathrm{Tr}_{\rho_1}\big(\mathrm{Hol}_{R_1}^{disc}(\check{A}^\perp + A_c^\perp, B)\big) d\check{\mu}_B^{\perp,disc}(\check{A}^\perp) = \mathrm{Tr}_{\rho_1}\big(\mathrm{Hol}_{R_1}^{disc}(A_c^\perp, B)\big) \tag{5.11}$$

By taking into account that $\prod_x 1_{\mathfrak{t}_{reg}}^{(s)}(B(x)) \neq 0$ for $B \in \mathcal{B}(\mathcal{K}) \subset \mathcal{B}(q\mathcal{K})$ implies $B \in \mathcal{B}_{reg}(q\mathcal{K})$ we then obtain from Eq. (5.11) and Eq. (3.29)

$$\mathrm{WLO}_{rig}^{disc}(L) = \sum_{y \in I} \int_\sim \Big\{ \big(\prod_x 1_{\mathfrak{t}_{reg}}^{(s)}(B(x))\big) \mathrm{Tr}_{\rho_1}\big(\mathrm{Hol}_{R_1}^{disc}(A_c^\perp, B)\big) \mathrm{Det}^{disc}(B) \Big\} \\ \times \exp\big(-2\pi i k \langle y, B(\sigma_0)\rangle\big) \exp(i S_{CS}^{disc}(A_c^\perp, B))(DA_c^\perp \otimes DB) \tag{5.12}$$

Proof of Eq. (5.11)*:* Let $A_c^\perp \in \mathcal{A}_c^\perp(K)$ and $B \in \mathcal{B}_{reg}(q\mathcal{K})$ be fixed. We will prove Eq. (5.11) by applying Proposition 4.4 to the special situation where

- $V = \check{\mathcal{A}}^\perp(K)$ and $d\mu = d\check{\mu}_B^{\perp,disc}$,
- $(Y_k^a)_{k \leq n, a \leq \dim(\mathfrak{g})}$ is the family of maps $Y_k^a : \check{\mathcal{A}}^\perp(K) \to \mathbb{R}$ given by

$$Y_k^a(\check{A}^\perp) = \big\langle e_a, \sum_{j=0}^2 w(j)\big(\check{A}^\perp(\bullet l_{S^1}^{j(k)})(l_\Sigma^{j(k)}) + A_c^\perp(l_\Sigma^{j(k)}) + B(\bullet l_\Sigma^{j(k)}) \cdot dt^{(N)}(l_{S^1}^{j(k)})\big)\big\rangle \tag{5.13}$$

  for all $\check{A}^\perp \in \check{\mathcal{A}}^\perp(K)$,
- $\Phi : \mathbb{R}^{n \times \dim(\mathfrak{g})} \to \mathbb{C}$ is given by

$$\Phi((x_k^a)_{k,a}) = \mathrm{Tr}_{\rho_1}\big(\prod_{k=1}^n \exp(\sum_{a=1}^{\dim(\mathfrak{g})} e_a x_k^a)\big) \quad \text{for all } (x_k^a)_{k,a} \in \mathbb{R}^{n \times \dim(\mathfrak{g})} \tag{5.14}$$

  Here $(e_a)_{a \leq \dim(\mathfrak{g})}$ is an arbitrary but fixed $\langle \cdot, \cdot \rangle$-orthonormal basis of $\mathfrak{g}$.

Note that $d\check{\mu}_B^{\perp,disc}$ is a well-defined normalized, non-degenerate, centered oscillatory Gauss-type measure. (Since by assumption $B \in \mathcal{B}_{reg}(q\mathcal{K})$ this follows from the remarks in Sec. 3.9). Moreover, we have

$$\mathrm{Tr}_{\rho_1}\big(\mathrm{Hol}_{R_1}^{disc}(\check{A}^\perp + A_c^\perp, B)\big) = \mathrm{Tr}_{\rho_1}\big(\prod_{k=1}^n \exp(\sum_{a=1}^{\dim(\mathfrak{g})} e_a Y_k^a(\check{A}^\perp))\big) \quad \forall \check{A}^\perp \in \check{\mathcal{A}}^\perp(K) \tag{5.15}$$

Finally, since $d\check{\mu}_B^{\perp,disc}$ is centered and normalized we have for every $k \leq n$ and $a \leq \dim(\mathfrak{g})$

$$\int_\sim Y_k^a \, d\check{\mu}_B^{\perp,disc} = Y_k^a(0) \tag{5.16}$$

Consequently, we obtain

$$\int_\sim \mathrm{Tr}_{\rho_1}\big(\mathrm{Hol}_{R_1}^{disc}(\check{A}^\perp + A_c^\perp, B)\big) d\check{\mu}_B^{\perp,disc}(\check{A}^\perp) \\ = \int_\sim \mathrm{Tr}_{\rho_1}\big(\prod_k \exp(\sum_a e_a Y_k^a)\big) d\check{\mu}_B^{\perp,disc} = \int_\sim \Phi((Y_k^a)_{k,a}) \, d\check{\mu}_B^{\perp,disc} \overset{(*)}{=} \Phi\big((\int_\sim Y_k^a \, d\check{\mu}_B^{\perp,disc})_{k,a}\big) \\ = \Phi((Y_k^a(0))_{k,a}) = \mathrm{Tr}_{\rho_1}\big(\prod_k \exp(\sum_a e_a Y_k^a(0))\big) = \mathrm{Tr}_{\rho_1}\big(\mathrm{Hol}_{R_1}^{disc}(A_c^\perp, B)\big) \tag{5.17}$$

where step $(*)$ follows from Proposition 4.4 above. The following remarks show that the assumptions of Proposition 4.4 are indeed fulfilled.

i) We have $\Phi \in \mathcal{P}_{exp}(\mathbb{R}^{n \times \dim(\mathfrak{g})})$. In order to see this note first that

$$\Phi((x_k^a)_{k,a}) = \operatorname{Tr}_{\rho_1}\big(\prod\nolimits_k \exp(\sum\nolimits_a e_a x_k^a)\big) = \operatorname{Tr}_{\operatorname{End}(V_1)}\big(\prod\nolimits_k \rho_1(\exp(\sum\nolimits_a e_a x_k^a))\big)$$
$$= \operatorname{Tr}_{\operatorname{End}(V_1)}\big(\prod\nolimits_k \exp_{\operatorname{End}(V_1)}(\sum\nolimits_a (\rho_1)_*(e_a) x_k^a)\big) \quad (5.18)$$

where V_1 is the representation space of ρ_1, $\exp_{\operatorname{End}(V_1)}$ is the exponential map of the associative algebra $\operatorname{End}(V_1)$, and $(\rho_1)_* : \mathfrak{g} \to \operatorname{gl}(V_1)$ is the Lie algebra representation induced by $\rho_1 : G \to \operatorname{GL}(V_1)$. Without loss of generality we can assume that $V_1 = \mathbb{C}^d$ where $d = \dim(V_1)$. From Definition 4.2 it now easily follows that we have indeed $\Phi \in \mathcal{P}_{exp}(\mathbb{R}^{n \times \dim(\mathfrak{g})})$.

ii) For all $k, k' \leq n$, $a, a' \leq \dim(\mathfrak{g})$ we have

$$\int_{\sim} Y_k^a Y_{k'}^{a'} \ d\check{\mu}_B^{\perp,disc} = \int_{\sim} Y_k^a \ d\check{\mu}_B^{\perp,disc} \int_{\sim} Y_{k'}^{a'} \ d\check{\mu}_B^{\perp,disc} \quad (5.19)$$

This follows from Eq. (5.16) above and

$$\int_{\sim} (Y_k^a - Y_k^a(0))(Y_{k'}^{a'} - Y_{k'}^{a'}(0)) d\check{\mu}_B^{\perp,disc} = \int_{\sim} \ll \cdot, f \gg \ll \cdot, f' \gg d\check{\mu}_B^{\perp,disc}$$
$$\overset{(*)}{\sim} \quad \ll f, \big(\star_K L^{(N)}(B)_{|\check{\mathcal{A}}^{\perp}(K)}\big)^{-1} f' \gg \overset{(**)}{=} 0 \quad (5.20)$$

where $\ll \cdot, \cdot \gg$ is the scalar product on[29] $\check{\mathcal{A}}^{\perp}(K)$ and, for given $k, k' \leq n$, $a, a' \leq \dim(\mathfrak{g})$, $f, f' \in \check{\mathcal{A}}^{\perp}(K)$ are chosen such that $Y_k^a(\check{A}^{\perp}) - Y_k^a(0) = \ll \check{A}^{\perp}, f \gg$ and $Y_{k'}^{a'}(\check{A}^{\perp}) - Y_{k'}^{a'}(0) = \ll \check{A}^{\perp}, f' \gg$ for all $\check{A}^{\perp} \in \check{\mathcal{A}}^{\perp}(K)$. Here in step $(*)$ we have used Proposition 4.1 (cf. also Remark 3.4), and in step $(**)$ we have used that for all non-trivial $l_{\Sigma}^{j(k)}$ and $l_{\Sigma}^{j'(k')}$, $k, k' \leq n$, $j, j' \in \{0, 1, 2\}$, appearing on the RHS of Eq. (5.13) we have

$$\star_K \pi(l_{\Sigma}^{j(k)}) \neq \pm \pi(l_{\Sigma}^{j'(k')}) \quad (5.21)$$

where $\pi : C^1(q\mathcal{K}, \mathbb{R}) \to C^1(K, \mathbb{R})$ is the real analogue of the orthogonal projection given in Eq. (3.4). Eq. (5.21) follows from Eq. (3.12) in Sec. 3.4 above and from our assumption that R_1 is a simplicial torus ribbon knot of standard type in $\mathcal{K} \times \mathbb{Z}_N$.

b) Step 2: Performing the $\int_{\sim} \cdots \exp(iS_{CS}^{disc}(A_c^{\perp}, B))(DA_c^{\perp} \otimes DB)$-integration in (5.12)

Note that the remaining fields $A_c^{\perp}$ and B in Eq. (5.12) take values in the Abelian Lie algebra $\mathfrak{t}$. For fixed $A_c^{\perp}$ and B we can therefore rewrite $\operatorname{Hol}_{R_1}^{disc}(A_c^{\perp}, B)$ as

$$\operatorname{Hol}_{R_1}^{disc}(A_c^{\perp}, B) = \exp\big(\Psi(B) + \sum\nolimits_{k=1}^{n} \sum\nolimits_{j=0}^{2} w(j) A_c^{\perp}(l_{\Sigma}^{j(k)})\big) \quad (5.22)$$

where we have set

$$\Psi(B) := \sum\nolimits_k \sum\nolimits_{j=0}^{2} w(j) B(\bullet\, l_{\Sigma}^{j(k)}) dt^{(N)}(l_{S^1}^{j(k)}) \quad \in \mathfrak{t} \quad (5.23)$$

Observation 5.10 From the assumption that $R = R_1$ is a simplicial torus ribbon knot in $\mathcal{K} \times \mathbb{Z}_N$ of standard type with first winding number $\mathbf{p} \neq 0$ it follows that for each $j = 0, 1, 2$ there is a simplicial loop $\mathfrak{l}_{\Sigma}^{j} = (\mathbf{x}_{\Sigma}^{j(k)})_{0 \leq k \leq \mathbf{n}}$, $\mathbf{n} \in \mathbb{N}$, in $q\mathcal{K}$ which, considered as a continuous map $S^1 \to S^2$, is an embedding and fulfills (cf. Convention 3.5)

$$\sum\nolimits_{k=1}^{n} l_{\Sigma}^{j(k)} = \mathbf{p} \sum\nolimits_{k=1}^{\mathbf{n}} \mathfrak{l}_{\Sigma}^{j(k)}$$

[29] which, according to Convention 3.2 above, is the the scalar product induced by $\ll \cdot, \cdot \gg_{\mathcal{A}^{\perp}(q\mathcal{K})}$

Since $\rho_1 = \rho_{\lambda_1}$ it follows from the definitions in Sec. 5.2 above that

$$\mathrm{Tr}_{\rho_1}(\exp(b)) = \sum\nolimits_{\alpha \in \Lambda} m_{\lambda_1}(\alpha) e^{2\pi i \langle \alpha, b\rangle} \quad \forall b \in \mathfrak{t} \tag{5.24}$$

Combining Eqs. (5.22) – (5.24) with Observation 5.10 we obtain

$$\begin{aligned} &\mathrm{Tr}_{\rho_1}\big(\mathrm{Hol}^{disc}_{R_1}(A^\perp_c, B)\big) \\ &= \sum\nolimits_{\alpha \in \Lambda} m_{\lambda_1}(\alpha)\big(\exp(2\pi i \langle \alpha, \Psi(B)\rangle)\big) \exp\big(2\pi i \ll A^\perp_c, \alpha\, \mathbf{p}\, (w\mathfrak{l})_\Sigma \gg_{\mathcal{A}^\perp(q\mathcal{K})}\big) \end{aligned} \tag{5.25}$$

where we have set

$$(w\mathfrak{l})_\Sigma := \sum\nolimits_{k=1}^{\mathbf{n}} \sum\nolimits_{j=0}^{2} w(j) \mathfrak{l}_\Sigma^{j(k)} \in C_1(q\mathcal{K}) \tag{5.26}$$

Let us now introduce for each $y \in I$, $\alpha \in \Lambda$ and $B \in \mathcal{B}(\mathcal{K}) \subset \mathcal{B}(q\mathcal{K})$:

$$F_{\alpha,y}(B) \; := \; \big(\prod\nolimits_x 1^{(s)}_{\mathfrak{t}_{reg}}(B(x))\big)\big(\exp(2\pi i \langle \alpha, \Psi(B)\rangle)\big)\, \mathrm{Det}^{disc}(B) \exp\big(-2\pi i k \langle y, B(\sigma_0)\rangle\big) \tag{5.27}$$

After doing so we can rewrite Eq. (5.12) as

$$\begin{aligned} \mathrm{WLO}^{disc}_{rig}(L) &= \sum\nolimits_{\alpha \in \Lambda} m_{\lambda_1}(\alpha) \sum\nolimits_{y \in I} \\ &\times \int_\sim F_{\alpha,y}(B) \exp\big(2\pi i \ll A^\perp_c, \alpha \mathbf{p}(w\mathfrak{l})_\Sigma \gg_{\mathcal{A}^\perp(q\mathcal{K})}\big) \exp(i S^{disc}_{CS}(A^\perp_c, B))(DA^\perp_c \otimes DB) \end{aligned} \tag{5.28}$$

For each fixed $y \in I$ and $\alpha \in \Lambda$ we will now evaluate the corresponding integral in Eq. (5.28) by applying Proposition 4.7 above in the special situation where

- $V = \mathcal{A}^\perp_c(K) \oplus \mathcal{B}(\mathcal{K})$. For V we use the decomposition $V = V_0 \oplus V_1 \oplus V_2$ given by

  $$\begin{aligned} V_0 &:= \mathcal{B}_c(q\mathcal{K}) \oplus (\mathrm{Image}(\star_K \circ \pi \circ d_{q\mathcal{K}}))^\perp \\ V_1 &:= (\ker(\star_K \circ \pi \circ d_{q\mathcal{K}}))^\perp = (\ker(\pi \circ d_{q\mathcal{K}}))^\perp \overset{(*)}{=} (\mathcal{B}_c(q\mathcal{K}))^\perp \subset \mathcal{B}(\mathcal{K}) \\ V_2 &:= \mathrm{Image}(\star_K \circ \pi \circ d_{q\mathcal{K}}) \subset \mathcal{A}^\perp_c(K) \end{aligned}$$

  where $\pi : \mathcal{A}_{\Sigma,\mathfrak{t}}(q\mathcal{K}) \to \mathcal{A}_{\Sigma,\mathfrak{t}}(K) \cong \mathcal{A}^\perp_c(K)$ is the orthogonal projection of Eq. (3.4), $d_{q\mathcal{K}}$ is a short notation for $(d_{q\mathcal{K}})_{|\mathcal{B}(\mathcal{K})}$, and $(\cdot)^\perp$ denotes the orthogonal complement in $\mathcal{A}^\perp_c(K)$ and $\mathcal{B}(\mathcal{K})$, respectively. Note that step $(*)$ follows from Eq. (3.5).

- $d\mu = d\nu^{disc} := \frac{1}{Z^{disc}} \exp(i S^{disc}_{CS}(A^\perp_c, B))(DA^\perp_c \otimes DB)$ where

  $$Z^{disc} := \int_\sim \exp(i S^{disc}_{CS}(A^\perp_c, B))(DA^\perp_c \otimes DB)$$

- $F = F_{\alpha,y} \circ p$ where $p : V_0 \oplus V_1 = \mathcal{B}(\mathcal{K}) \oplus (V_2)^\perp \to \mathcal{B}(\mathcal{K})$ is the canonical projection.

- $v = 2\pi \alpha \mathbf{p}(w\mathfrak{l})_\Sigma$

Before we continue we need to verify that the assumptions of Proposition 4.7 above are indeed fulfilled.

1. $d\nu^{disc}$ is a normalized, centered oscillatory Gauss type measure on $\mathcal{A}^\perp_c(K) \oplus \mathcal{B}(q\mathcal{K})$. In order to see this we rewrite $d\nu^{disc}$ as[30]

   $$d\nu^{disc} = \frac{1}{Z^{disc}} \exp(i \ll A^\perp_c, -2\pi k(\star_K \circ \pi \circ d_{q\mathcal{K}})B \gg_{\mathcal{A}^\perp(q\mathcal{K})})(DA^\perp_c \otimes DB)$$

   Accordingly, $d\nu^{disc}$ has the form as in Proposition 4.7 with V_0, V_1, and V_2 as above and where $M : V_1 \to V_2$ is the well-defined linear isomorphism given by

   $$M = -2\pi k(\star_K \circ \pi \circ d_{q\mathcal{K}})_{|V_1} \tag{5.29}$$

[30] recall Eq. (3.15c) and observe that $\ll \star_K A^\perp_c, d_{q\mathcal{K}} B \gg_{\mathcal{A}^\perp(q\mathcal{K})} = \ll \star_K A^\perp_c, \pi(d_{q\mathcal{K}} B) \gg_{\mathcal{A}^\perp(q\mathcal{K})} = - \ll A^\perp_c, \star_K(\pi(d_{q\mathcal{K}} B)) \gg_{\mathcal{A}^\perp(q\mathcal{K})}$

2. $F = F_{\alpha,y} \circ p$ is a bounded and uniformly continuous function.

3. v is an element of V_2. In order to prove this we introduce a linear map

$$m_{\mathbb{R}} : C^0(q\mathcal{K}, \mathbb{R}) \to C^1(K, \mathbb{R})$$

by $m_{\mathbb{R}} := \star_K \circ \pi \circ d_{q\mathcal{K}}$ where $\pi : C^1(q\mathcal{K}, \mathbb{R}) \to C^1(K, \mathbb{R})$, $d_{q\mathcal{K}} : C^0(q\mathcal{K}, \mathbb{R}) \to C^1(q\mathcal{K}, \mathbb{R})$, and $\star_K : C^1(K, \mathbb{R}) \to C^1(K, \mathbb{R})$ are the "real analogues" of the three maps appearing on the RHS of Eq. (5.29) above. From Lemma 5.11 in Step 3 below it follows that there is a unique[31] $f \in C^0(\mathcal{K}, \mathbb{R}) \subset C^0(q\mathcal{K}, \mathbb{R})$ such that

$$(w\mathfrak{l})_\Sigma = m_{\mathbb{R}} \cdot f \tag{5.30}$$

and also

$$\sum\nolimits_{x \in \mathfrak{F}_0(q\mathcal{K})} f(x) = 0 \tag{5.31}$$

That $v \in V_2$ now follows from

$$v = 2\pi\alpha\mathbf{p}(w\mathfrak{l})_\Sigma = 2\pi\alpha\mathbf{p}(m_{\mathbb{R}} \cdot f) = (\star_K \circ \pi \circ d_{q\mathcal{K}}) \cdot \big(2\pi\alpha\mathbf{p}f\big) \tag{5.32}$$

By applying Proposition 4.7 we now obtain

$$\begin{aligned}
&\tfrac{1}{Z^{disc}} \int_\sim F_{\alpha,y}(B) \exp\big(2\pi i \ll A_c^\perp, \alpha\mathbf{p}(w\mathfrak{l})_\Sigma \gg_{\mathcal{A}^\perp(q\mathcal{K})}\big) \exp(iS^{disc}_{CS}(A_c^\perp, B))(DA_c^\perp \otimes DB) \\
&= \int_\sim F(B) \exp\big(i \ll A_c^\perp, v \gg_{\mathcal{A}^\perp(q\mathcal{K})}\big) d\nu^{disc}(A_c^\perp, B) \\
&\quad \sim \int^\sim_{V_0} F(x_0 - M^{-1}v) dx_0 \overset{(*)}{\sim} \int^\sim_{\mathfrak{t}} F_{\alpha,y}(b - M^{-1}v) db
\end{aligned} \tag{5.33}$$

Here $\int^\sim \cdots dx_0$ and $\int^\sim \cdots db$ are improper integrals defined according to Definition 4.5 above. (Remark 4.6 above and a "periodicity argument", which will be given in Step 4 below, imply that these improper integrals are well-defined.) Step $(*)$ above follows because $F(x_0 - M^{-1}v)$ depends only on the $\mathcal{B}_c(q\mathcal{K})$-component of $x_0 \in V_0 = \mathcal{B}_c(q\mathcal{K}) \oplus (V_2)^\perp \cong \mathfrak{t} \oplus (V_2)^\perp$.

According to Eq. (5.31) we have $\alpha f \in V_1 = (\mathcal{B}_c(q\mathcal{K}))^\perp$, so Eq. (5.32) implies that

$$M^{-1}v = -\tfrac{1}{k}\alpha\mathbf{p}f \tag{5.34}$$

Combining Eqs. (5.27), (5.28), (5.33), and (5.34) we obtain

$$\begin{aligned}
\mathrm{WLO}^{disc}_{rig}(L) \sim \sum\nolimits_{\alpha \in \Lambda} m_{\lambda_1}(\alpha) \sum\nolimits_{y \in I} \\
\times \int^\sim_{\mathfrak{t}} db \, \Big[\exp\big(-2\pi i k \langle y, B(\sigma_0) \rangle\big) \big(\prod\nolimits_x 1^{(s)}_{\mathfrak{t}_{reg}}(B(x))\big) \\
\times \big(\exp(2\pi i \langle \alpha, \Psi(B) \rangle)\big) \, \mathrm{Det}^{disc}(B) \Big]_{|B = b + \frac{1}{k}\alpha\mathbf{p}f}
\end{aligned} \tag{5.35}$$

It is not difficult to see that Eq. (5.35) also holds if we redefine f using the normalization condition

$$f(\sigma_0) = 0 \tag{5.36}$$

instead of the normalization condition (5.31) above[32].

[31] here $C^0(\mathcal{K}, \mathbb{R})$ is embedded into $C^0(q\mathcal{K}, \mathbb{R})$ in the same way as $\mathcal{B}(\mathcal{K})$ is embedded into $\mathcal{B}(q\mathcal{K})$, cf. Sec. 3.2

[32] this follows from Eq. (4.6) in Remark 4.6 above and the periodicity properties of the integrand in $\int^\sim_{\mathfrak{t}} \cdots db$ (for fixed y and α), cf. Step 4 below

c) Step 3: Rewriting Eq. (5.35)

Lemma 5.11 *Assume that* $(w\mathfrak{l})_\Sigma \in C_1(q\mathcal{K}) \cong C^1(q\mathcal{K}, \mathbb{R})$ *is as in Eq.* (5.26) *above. Then there is a* $f \in C^0(\mathcal{K}, \mathbb{R}) \subset C^0(q\mathcal{K}, \mathbb{R})$*, unique up to an additive constant, such that* $(w\mathfrak{l})_\Sigma = m_\mathbb{R} \cdot f$*.*

Proof. That f is unique up to an additive constant follows by combining the definition of $m_\mathbb{R}$ with the real analogue of Eq. (3.5) in Sec. 3.2 above and the fact that $\star_K : C^1(K, \mathbb{R}) \to C^1(K, \mathbb{R})$ is a bijection.

In order to show the existence of f we observe first that the assumption that $R = R_1$ is a simplicial torus ribbon knot of standard type in $\mathcal{K} \times \mathbb{Z}_N$ implies that $\Sigma \backslash (\text{arc}(\mathfrak{l}^1_\Sigma) \cup \text{arc}(\mathfrak{l}^2_\Sigma))$ has three connected components. Let us denote these three connected components by Z_0, Z_1, Z_2. The enumeration is chosen such that Z_0 is the connected component containing $\text{arc}(\mathfrak{l}^0_\Sigma)$ and Z_1 is the other connected component having $\text{arc}(\mathfrak{l}^1_\Sigma)$ on its boundary. Let $f : \mathfrak{F}_0(\mathcal{K}) \to \mathbb{R}$ be given by

$$f(\sigma) := \begin{cases} c & \text{if } \sigma \in \overline{Z_1} \\ c \pm \frac{1}{2} & \text{if } \sigma \in Z_0 \\ c \pm 1 & \text{if } \sigma \in \overline{Z_2} \end{cases} \tag{5.37}$$

for all $\sigma \in \mathfrak{F}_0(\mathcal{K})$ where $\overline{Z_i}$ is the closure of Z_i in Σ, $c \in \mathbb{R}$ is an arbitrary constant, and the sign $\pm$ is "+" if for any $k \leq \mathbf{n}$ the edge $\star_K(\pi(\mathfrak{l}^{0(k)}_\Sigma))$ points from Z_2 to Z_1 and "$-$" otherwise. In order to conclude the proof of Lemma 5.11 we have to show that

$$(w\mathfrak{l})_\Sigma = m_\mathbb{R} \cdot f = \star_K(\pi(d_{q\mathcal{K}} f)) \tag{5.38}$$

Let S be the set of those $e \in \mathfrak{F}_1(q\mathcal{K})$ which are contained in Z_0 but do not lie on $\text{arc}(\mathfrak{l}^0_\Sigma)$. Now observe that $(d_{q\mathcal{K}} f)(e) = 0$ if $e \notin S$. On the other hand if $e \in S$ we have

$$(d_{q\mathcal{K}} f)(e) = \pm\tfrac{1}{2}\, \text{sgn}(e)$$

where the sign $\pm$ is the same as in Eq. (5.37) and where $\text{sgn}(e) = 1$ if the (oriented) edge e "points from" the region Z_1 to the region Z_2 and $\text{sgn}(e) = -1$ otherwise.

From the definition of $q\mathcal{K}$ it follows that every $e \in \mathfrak{F}_1(q\mathcal{K})$ has exactly one endpoint in $\mathfrak{F}_0(K_1) \cup \mathfrak{F}_0(K_2) \subset \mathfrak{F}_0(q\mathcal{K})$ and one endpoint in $(\mathfrak{F}_0(K_1) \cup \mathfrak{F}_0(K_2))^c := \mathfrak{F}_0(q\mathcal{K}) \backslash (\mathfrak{F}_0(K_1) \cup \mathfrak{F}_0(K_2))$. On the other hand, if $e \in S$ then both endpoints of e will be in $\bigcup_{j=0}^2 \text{arc}(\mathfrak{l}^j_\Sigma)$. Accordingly, for every $e \in S$ we can distinguish between exactly three types:

$$e \text{ is of type } j \ (j = 0, 1, 2) \text{ if } e \text{ has an endpoint in } \text{arc}(\mathfrak{l}^j_\Sigma) \cap (\mathfrak{F}_0(K_1) \cup \mathfrak{F}_0(K_2))^c$$

Next we observe that for each fixed $e \in S$ of type j there exist exactly two indices $k \leq \mathbf{n}$ such that

$$\star_K\left(\pi(\text{sgn}(e) \cdot e)\right) = \pi(\mathfrak{l}^{j(k)}_\Sigma) \tag{5.39}$$

Moreover, if we let $e \in S$ vary then for $j = 1, 2$ every $k \leq \mathbf{n}$ arises exactly once on the RHS of Eq. (5.39) and for $j = 0$ every $k \leq \mathbf{n}$ arises exactly twice Taking this into account we arrive at

$$\begin{aligned} \star_K\left(\pi(d_{q\mathcal{K}} f)\right) &= \sum\nolimits_{e \in S} \tfrac{1}{2} \star_K\left(\pi(\text{sgn}(e) \cdot e)\right) = \sum\nolimits_{k=1}^{\mathbf{n}} \left(\tfrac{1}{4}\pi(\mathfrak{l}^{1(k)}_\Sigma) + \tfrac{1}{4}\pi(\mathfrak{l}^{2(k)}_\Sigma) + 2\tfrac{1}{4}\pi(\mathfrak{l}^{0(k)}_\Sigma)\right) \\ &= \sum\nolimits_{k=1}^{\mathbf{n}} \left(\tfrac{1}{4}\mathfrak{l}^{1(k)}_\Sigma + \tfrac{1}{4}\mathfrak{l}^{2(k)}_\Sigma + \tfrac{1}{2}\mathfrak{l}^{0(k)}_\Sigma\right) = \sum\nolimits_{k=1}^{\mathbf{n}} \sum\nolimits_{j=0}^{2} w(j)\mathfrak{l}^{j(k)}_\Sigma = (w\mathfrak{l})_\Sigma \end{aligned}$$

□

In the following we assume that $B \in \mathcal{B}(\mathcal{K}) \subset \mathcal{B}(q\mathcal{K}) = C^0(q\mathcal{K}, \mathfrak{t})$ is of the form

$$B = b + \tfrac{1}{k}\alpha \mathbf{p} f \tag{5.40}$$

with $b \in \mathfrak{t}$, $\alpha \in \Lambda$ and where f is given by Eq. (5.30) above in combination with (5.36).

Observation 5.12 *Let Z_0, Z_1, Z_2 be as in the proof of Lemma 5.11 above. Then the restriction of $B : \mathfrak{F}_0(q\mathcal{K}) \to \mathfrak{t}$ to $\mathfrak{F}_0(q\mathcal{K}) \cap Z_i$ is constant for each $i = 0, 1, 2$. Moreover, if $i = 1, 2$ then also the restriction of $B : \mathfrak{F}_0(q\mathcal{K}) \to \mathfrak{t}$ to $\mathfrak{F}_0(q\mathcal{K}) \cap \overline{Z_i}$ is constant.*

We set $B(Z_i) := B(x)$ for any $x \in Z_i \cap \mathfrak{F}_0(q\mathcal{K})$, which – according to Observation 5.12 – is well-defined.

Lemma 5.13 *For every $B \in \mathcal{B}(\mathcal{K})$ of the form (5.40) and fulfilling $\prod_x 1^{(s)}_{\mathfrak{t}_{reg}}(B(x)) \neq 0$ we have*[33]

$$\mathrm{Det}^{disc}(B) = \prod_{i=0}^{2} \det{}^{1/2}\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B(Z_i)))_{|\mathfrak{k}}\big)^{\chi(Z_i)} \tag{5.41}$$

where Z_i, $i = 0, 1, 2$ are as in the proof of Lemma 5.11 and $\chi(Z_i)$ is the Euler characteristic of Z_i.

Proof. We set $\mathfrak{F}_p(\overline{Z_i}) := \{F \in \mathfrak{F}_p(\mathcal{K}) \mid \bar{F} \in \overline{Z_i}\} = \{F \in \mathfrak{F}_p(\mathcal{K}) \mid F \subset \overline{Z_i}\}$, for $i = 0, 1, 2$, and $\mathfrak{F}_p(Z_0) := \{F \in \mathfrak{F}_p(\mathcal{K}) \mid \bar{F} \in Z_0\}$. Since $\Sigma = S^2$ is the disjoint union of the three sets Z_0, $\overline{Z_1}$, and $\overline{Z_2}$ we obtain from Eq. (3.23) in Sec. 3.6 and Observation 5.12

$$\begin{aligned}\mathrm{Det}^{disc}(B) &= \prod_{p=0}^{2} \Bigg[\Big(\prod_{i=1}^{2} \det{}^{1/2}\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B(Z_i)))_{|\mathfrak{k}}\big)^{(-1)^p} \Big)^{\#\mathfrak{F}_p(\overline{Z_i})} \\ &\qquad \times \Big(\det{}^{1/2}\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B(Z_0)))_{|\mathfrak{k}}\big)^{(-1)^p} \Big)^{\#\mathfrak{F}_p(Z_0)} \Bigg] \\ &\overset{(*)}{=} \prod_{i=0}^{2} \det{}^{1/2}\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B(Z_i)))_{|\mathfrak{k}}\big)^{\sum_{p=0}^{2}(-1)^p \#\mathfrak{F}_p(\overline{Z_i})}\end{aligned} \tag{5.42}$$

where in step $(*)$ we have used that $\sum_{p=0}^{2}(-1)^p \#\mathfrak{F}_p(Z_0) = \sum_{p=0}^{2}(-1)^p \#\mathfrak{F}_p(\overline{Z_0})$. The assertion of the lemma now follows by combining Eq. (5.42) with

$$\chi(Z_i) = \chi(\overline{Z_i}) = \sum_{p=0}^{2} (-1)^p \#\mathfrak{F}_p(\overline{Z_i}), \quad i = 0, 1, 2$$

where we have used that $\overline{Z_i}$ is a subcomplex of the CW complex $\mathcal{K}$.

□

Taking into account that $\mathrm{arc}(l^j_\Sigma) \subset \overline{Z_j}$ for $j = 1, 2$ and $\mathrm{arc}(l^0_\Sigma) \subset Z_0$ we see that Observation 5.12 implies that

$$B(Z_j) = B(\bullet\, l^{j(k)}_\Sigma) \quad \forall k \leq n \tag{5.43}$$

for every B of the form in Eq. (5.40). Moreover, for such B we have

$$B(Z_0) = \tfrac{1}{2}(B(Z_1) + B(Z_2)) \tag{5.44}$$

Finally, note that for $j = 0, 1, 2$ we have

$$\mathbf{q} = \mathrm{wind}(l^j_{S^1}) = \sum_k dt^{(N)}(l^{j(k)}_{S^1}) \tag{5.45}$$

In order to see this recall that $\mathbf{q}$ is the second winding number of the simplicial torus ribbon knot of standard type R_1, which coincides with the winding number $\mathrm{wind}(l^j_{S^1})$ of $l^j_{S^1}$, considered as a continuous map $S^1 \to S^1$.

[33] Note that the expression on the RHS of Eq. (5.41) is well-defined since by assumption $\prod_x 1^{(s)}_{\mathfrak{t}_{reg}}(B(x)) \neq 0$, which implies that $B(x) \in \mathfrak{t}_{reg}$ and therefore $\det\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B(x)))_{|\mathfrak{k}}\big) \neq 0$ (recall the definition of $\mathfrak{t}_{reg}$ and cf. Eq (3.22) above)

Combining Eq. (5.35) and Eq. (5.23) with Eqs. (5.43) – (5.45) and Lemma 5.13, and taking into account Eq. (5.36) above (cf. the paragraph before Observation 5.12)

$$\mathrm{WLO}^{disc}_{rig}(L) \sim \sum\nolimits_{\alpha\in\Lambda} m_{\lambda_1}(\alpha) \ \sum\nolimits_{y\in I} \int_{\mathfrak{t}}^{\sim} db\ e^{-2\pi i k\langle y,b\rangle} F_\alpha(b) \tag{5.46}$$

where for $b \in \mathfrak{t}$ and $\alpha \in \Lambda$ we have set

$$F_\alpha(b) := \Big[\big(\prod\nolimits_x 1^{(s)}_{\mathfrak{t}_{reg}}(B(x))\big)\big(\exp(\pi i \mathbf{q}\langle \alpha, B(Z_1)+B(Z_2)\rangle\big) \\ \times \prod\nolimits_{i=0}^{2} \det{}^{1/2}\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B(Z_i)))_{|\mathfrak{k}}\big)^{\chi(Z_i)}\big)\Big]_{|B=b+\frac{1}{k}\alpha\mathbf{p}f} \tag{5.47}$$

d) Step 4: Performing $\int^{\sim}\cdots db$ and $\sum_{y\in I}$ in Eq. (5.46)

For all $y \in I$ and $\alpha \in \Lambda$ the function $\mathfrak{t} \ni b \mapsto e^{-2\pi i k\langle y,b\rangle}F_\alpha(b) \in \mathbb{C}$ is invariant under all translations of the form $b \mapsto b + x$ where $x \in I = \ker(\exp_{|\mathfrak{t}}) \cong \mathbb{Z}^{\dim(\mathfrak{t})}$. In order to prove this it is enough to show that for all $b \in \mathfrak{t}$ and $x \in I$ we have

$$1^{(s)}_{\mathfrak{t}_{reg}}(b+x) = 1^{(s)}_{\mathfrak{t}_{reg}}(b) \tag{5.48a}$$
$$e^{2\pi i\epsilon\langle\alpha,b+x\rangle} = e^{2\pi i\epsilon\langle\alpha,b\rangle} \quad \text{for all } \alpha\in\Lambda,\ \epsilon\in\mathbb{Z} \tag{5.48b}$$
$$\det{}^{1/2}(1_{\mathfrak{k}} - \exp(\mathrm{ad}(b+x))_{|\mathfrak{k}}) = \det{}^{1/2}(1_{\mathfrak{k}} - \exp(\mathrm{ad}(b))_{|\mathfrak{k}}) \tag{5.48c}$$
$$e^{-2\pi i k\langle y,b+x\rangle} = e^{-2\pi i k\langle y,b\rangle} \quad \text{for all } y\in I \tag{5.48d}$$

Note that because of the assumption that G is simply-connected we have $\Gamma = I$. The first of the four equations above therefore follows from the assumption in Sec. 3.7 that $1^{(s)}_{\mathfrak{t}_{reg}}$ is invariant under $\mathcal{W}_{\mathrm{aff}}$. The second equation follows because by definition, Λ is the lattice dual to $\Gamma = I$. The third equation follows from Eq. (3.22) above by taking into account that $(-1)^{\sum_{\alpha\in\mathcal{R}_+}\langle\alpha,x\rangle} = (-1)^{2\langle\rho,x\rangle} = 1$ for $x \in \Gamma = I$ because $\rho = \frac{1}{2}\sum_{\alpha\in\mathcal{R}_+}$ is an element of the weight lattice Λ. Finally, in order to see that the fourth equation holds, observe that it is enough to show that

$$\langle\check{\alpha},\check{\beta}\rangle \in \mathbb{Z} \quad \text{for all coroots } \check{\alpha},\check{\beta} \tag{5.49}$$

According to the general theory of semi-simple Lie algebras we have $2\frac{\langle\check{\alpha},\check{\beta}\rangle}{\langle\check{\alpha},\check{\alpha}\rangle} \in \mathbb{Z}$. Moreover, there are at most two different (co)roots lengths and the quotient between the square lengths of the long and short coroots is either 1, 2, or 3. Since the normalization of $\langle\cdot,\cdot\rangle$ was chosen such that $\langle\check{\alpha},\check{\alpha}\rangle = 2$ holds if $\check{\alpha}$ is a short coroot we therefore have $\langle\check{\alpha},\check{\alpha}\rangle/2 \in \{1,2,3\}$ and (5.49) follows.

From Eqs. (5.47) and (5.48) we conclude that $\mathfrak{t} \ni b \mapsto e^{-2\pi i k\langle y,b\rangle}F_\alpha(b) \in \mathbb{C}$ is indeed I-periodic and we can therefore apply Eq. (4.5) in Remark 4.6 above and obtain

$$\int^{\sim} db\ e^{-2\pi i k\langle y,b\rangle}F_\alpha(b) \sim \int_Q db\ e^{-2\pi i k\langle y,b\rangle}F_\alpha(b) = \int db\ e^{-2\pi i k\langle y,b\rangle}1_Q(b)F_\alpha(b) \tag{5.50}$$

where on the RHS $\int_Q\cdots db$ and $\int\cdots db$ are now ordinary integrals and where we have set

$$Q := \{\sum\nolimits_i \lambda_i e_i \mid \lambda_i \in (0,1) \text{ for all } i \le m\} \subset \mathfrak{t}, \tag{5.51}$$

Here $(e_i)_{i\le m}$ is an (arbitrary) fixed basis of I.

According to Eq. (5.50) we can now rewrite Eq. (5.46) as

$$\mathrm{WLO}^{disc}_{rig}(L) \sim \sum\nolimits_{\alpha\in\Lambda} m_{\lambda_1}(\alpha)\sum\nolimits_{y\in I}\int db\ e^{-2\pi i k\langle y,b\rangle}1_Q(b)F_\alpha(b) \\ \overset{(*)}{\sim} \sum\nolimits_{\alpha\in\Lambda} m_{\lambda_1}(\alpha)\sum\nolimits_{b\in\frac{1}{k}\Lambda} 1_Q(b)F_\alpha(b) \tag{5.52}$$

where in step $(*)$ we have used, for each $\alpha \in \Lambda$, the Poisson summation formula

$$\sum\nolimits_{y \in I} e^{-2\pi i k \langle y, b\rangle} = c_\Lambda \sum\nolimits_{x \in \frac{1}{k}\Lambda} \delta_x(b) \tag{5.53}$$

where δ_x is the delta distribution in $x \in \mathfrak{t}$ and c_Λ a constant depending on the lattice Λ. (Recall that the lattice Λ is dual to $\Gamma = I$.) Observe also that $1_Q F_\alpha$ clearly has compact support and that $1_Q F_\alpha$ is smooth because $\partial Q \subset \mathfrak{t}_{sing} = \mathfrak{t} \backslash \mathfrak{t}_{reg}$ and F_α vanishes[34] on a neighborhood of $\mathfrak{t}_{sing}$.

Finally, note that since $s > 0$ above was chosen small enough (cf. Footnote 13 in Sec. 3.7) we have for every $B \in \mathcal{B}(q\mathcal{K})$ of the form Eq. (5.40) with $b \in \frac{1}{k}\Lambda$

$$\prod\nolimits_{x \in \mathfrak{F}_0(q\mathcal{K})} 1^{(s)}_{\mathfrak{t}_{reg}}(B(x)) = \prod\nolimits_{x \in \mathfrak{F}_0(q\mathcal{K})} 1_{\mathfrak{t}_{reg}}(B(x)) \overset{(+)}{=} \prod\nolimits_{i=0}^{2} 1_{\mathfrak{t}_{reg}}(B(Z_i)) \tag{5.54}$$

where step $(+)$ follows from Observation 5.12.

Using this we obtain from Eq. (5.52) and Eq. (5.47) after the change of variable $b \to kb =: \alpha_0$ and writing α_1 instead of α (and by taking into account that $\chi(Z_0) = 0$ and $\chi(Z_1) = \chi(Z_2) = 1$):

$$\begin{aligned} \mathrm{WLO}^{disc}_{rig}(L) \sim \sum\nolimits_{\alpha_0, \alpha_1 \in \Lambda} & 1_{kQ}(\alpha_0) m_{\lambda_1}(\alpha_1) \\ & \times \Bigg[\Big(\prod\nolimits_{i=0}^{2} 1_{\mathfrak{t}_{reg}}(B(Z_i)) \Big) \Big(\prod\nolimits_{i=1}^{2} \det\nolimits^{1/2}\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B(Z_i)))_{|\mathfrak{k}}\big) \Big) \\ & \times \exp(\pi i \mathbf{q} \langle \alpha_1, B(Z_1) + B(Z_2) \rangle) \Bigg]_{|B = \frac{1}{k}(\alpha_0 + \alpha_1 \mathbf{p} f)} \end{aligned} \tag{5.55}$$

Recall from the paragraph at the beginning of Sec. 5.4 that so far we have been working with the original definition of $\mathrm{WLO}^{disc}_{rig}(L)$ given in Sec. 3.10.

By examining the calculations above it it becomes clear[35] that if one modifies the definition of $\mathrm{WLO}^{disc}_{rig}(L)$ in one of the possible ways listed in Sec. 3.11 then instead of Eq. (5.55) one arrives at

$$\begin{aligned} \mathrm{WLO}^{disc}_{rig}(L) \sim \sum\nolimits_{\alpha_0, \alpha_1 \in \Lambda} & 1_{kQ}(\alpha_0) m_{\lambda_1}(\alpha_1) \\ & \times \Big(\prod\nolimits_{i=1}^{2} 1_{\mathfrak{t}_{reg}}(B(Z_i)) \Big) \Big(\prod\nolimits_{i=1}^{2} \det\nolimits^{1/2}\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B(Z_i)))_{|\mathfrak{k}}\big) \Big) \\ & \times \exp(\pi i \mathbf{q} \langle \alpha_1, B(Z_1) + B(Z_2) \rangle) \Big]_{|B = \frac{1}{k}(\alpha_0 + \alpha_1 \mathbf{p} f)} \end{aligned} \tag{5.56}$$

Note that the only difference between Eq. (5.56) and (5.55) is that the $1_{\mathfrak{t}_{reg}}(B(Z_0))$-factor appearing in Eq. (5.55) no longer appears in Eq. (5.56).

e) Step 5: Some algebraic/combinatorial arguments

For each $\alpha_0, \alpha_1 \in \Lambda$ we define

$$\eta_{(\alpha_0, \alpha_1)} : \{1, 2\} \to \Lambda$$

by

$$\eta_{(\alpha_0, \alpha_1)}(i) = kB(Z_i) - \rho \qquad i = 1, 2 \tag{5.57}$$

where $B = \frac{1}{k}(\alpha_0 + \alpha_1 \mathbf{p} f)$. Observe that for $\eta = \eta_{(\alpha_0, \alpha_1)}$ and $B = \frac{1}{k}(\alpha_0 + \alpha_1 \mathbf{p} f)$ we have

$$\det\nolimits^{1/2}\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(B(Z_i)))_{|\mathfrak{k}}\big) = \det\nolimits^{1/2}\big(1_{\mathfrak{k}} - \exp(\mathrm{ad}(\tfrac{1}{k}(\eta(i) + \rho))_{|\mathfrak{k}})\big) \overset{(*)}{\sim} d_{\eta(i)}, \tag{5.58a}$$

[34]note that according to the definition of F_α and Eq. (5.36) there is a factor $1^{(s)}_{\mathfrak{t}_{reg}}(b)$ appearing in $F_\alpha(b)$ which vanishes on a neighborhood of $\mathfrak{t}_{sing}$, cf. Sec. 3.7

[35]for example, if one works with the first modification (M1) on the list in Sec. 3.11 then this point is obvious after examining the proof of Theorem 3.5 in [16] where simplicial ribbons in $q\mathcal{K} \times \mathbb{Z}_N$ are used. For modification (M2) this is also not difficult to see

where in step $(*)$ we used Eq. (3.22) and Eq. (5.3).

In the following we will assume, without loss of generality[36] that $\sigma_0 \in Z_1$ and therefore

$$B(\sigma_0) = B(Z_1) \tag{5.58b}$$

Moreover, we will assume, without loss of generality[37] that the enumeration of Z_1 and Z_2 was chosen such that we have the situation where in Eq. (5.37) in the proof of Lemma 5.11 the "$-$"-sign appears. Then we have

$$\alpha_0 = kB(\sigma_0) = kB(Z_1) = \eta(1) + \rho, \tag{5.58c}$$

$$\alpha_1 = \tfrac{1}{\mathbf{p}}(\eta(1) - \eta(2)) \tag{5.58d}$$

$$\begin{aligned}\exp(\pi i \mathbf{q}\langle \alpha_1, B(Z_1) + B(Z_2)\rangle) &= \exp(\pi i \mathbf{q}\langle \tfrac{1}{\mathbf{p}}(\eta(1) - \eta(2)), \tfrac{1}{k}(\eta(1) + \eta(2) + 2\rho)\rangle \\ &= \exp\big(\tfrac{\pi i}{k}\tfrac{\mathbf{q}}{\mathbf{p}}\langle \eta(1), \eta(1) + 2\rho\rangle - \langle \eta(2), \eta(2) + 2\rho\rangle\big) = \theta_{\eta(1)}^{\frac{\mathbf{q}}{\mathbf{p}}} \theta_{\eta(2)}^{-\frac{\mathbf{q}}{\mathbf{p}}}\end{aligned} \tag{5.58e}$$

In view of the previous equations it is clear that we can rewrite Eq. (5.56) in the following form

$$\begin{aligned}\mathrm{WLO}_{rig}^{disc}(L) \sim \sum\nolimits_{\alpha_0, \alpha_1 \in \Lambda} \Big[& m_{\lambda_1}\big(\tfrac{1}{\mathbf{p}}(\eta(1) - \eta(2))\big) 1_{kQ}(\eta(1) + \rho) \big(\prod\nolimits_{i=1}^{2} 1_{\mathfrak{t}_{reg}}(\tfrac{1}{k}(\eta(i) + \rho))\big) \\ & \times d_{\eta(1)} d_{\eta(2)} \theta_{\eta(1)}^{\frac{\mathbf{q}}{\mathbf{p}}} \theta_{\eta(2)}^{-\frac{\mathbf{q}}{\mathbf{p}}} \Big]_{|\eta = \eta_{(\alpha_0, \alpha_1)}}\end{aligned} \tag{5.59}$$

But

$$\begin{aligned}& 1_{kQ}(\eta(1) + \rho) \ \big(\prod\nolimits_{i=1}^{2} 1_{\mathfrak{t}_{reg}}(\tfrac{1}{k}(\eta(i) + \rho))\big) \\ &\quad = 1_{kQ}(\eta(1) + \rho) 1_{k\mathfrak{t}_{reg}}(\eta(1) + \rho) 1_{k\mathfrak{t}_{reg}}(\eta(2) + \rho) = 1_{k(Q \cap \mathfrak{t}_{reg})}(\eta(1) + \rho) 1_{k\mathfrak{t}_{reg}}(\eta(2) + \rho)\end{aligned}$$

Combining this with Eq. (5.59) we obtain

$$\begin{aligned}\mathrm{WLO}_{rig}^{disc}(L) &\sim \sum\nolimits_{\eta_1, \eta_2 \in \Lambda} \Big[m_{\lambda_1}\big(\tfrac{1}{\mathbf{p}}(\eta_1 - \eta_2)\big) 1_{k(Q \cap \mathfrak{t}_{reg})}(\eta_1 + \rho) 1_{k\mathfrak{t}_{reg}}(\eta_2 + \rho) d_{\eta_1} d_{\eta_2} \theta_{\eta_1}^{\frac{\mathbf{q}}{\mathbf{p}}} \theta_{\eta_2}^{-\frac{\mathbf{q}}{\mathbf{p}}} \Big] \\ &\sim \sum\nolimits_{\eta_1 \in k(Q \cap \mathfrak{t}_{reg} - \rho) \cap \Lambda, \eta_2 \in (k\mathfrak{t}_{reg} - \rho) \cap \Lambda} \Big[m_{\lambda_1}\big(\tfrac{1}{\mathbf{p}}(\eta_1 - \eta_2)\big) d_{\eta_1} d_{\eta_2} \theta_{\eta_1}^{\frac{\mathbf{q}}{\mathbf{p}}} \theta_{\eta_2}^{-\frac{\mathbf{q}}{\mathbf{p}}} \Big]\end{aligned} \tag{5.60}$$

Let P be the unique Weyl alcove which is contained in the Weyl chamber $\mathcal{C}$ fixed above and which has $0 \in \mathfrak{t}$ on its boundary. Explicitly, P is given by

$$P = \{b \in \mathcal{C} \mid \langle b, \theta \rangle < 1\} \tag{5.61}$$

Note that the map

$$\mathcal{W}_{\text{aff}} \times P \ni (\tau, b) \mapsto \tau \cdot b \in \mathfrak{t}_{reg} \tag{5.62a}$$

is a well-defined bijection. Moreover, there is a finite subset W of $\mathcal{W}_{\text{aff}}$ such that

$$W \times P \ni (\tau, b) \mapsto \tau \cdot b \in Q \cap \mathfrak{t}_{reg} \tag{5.62b}$$

is a bijection, too. Clearly, these two bijections above induce two other bijections

$$\mathcal{W}_{\text{aff}} \times (kP - \rho) \ni (\tau, b) \mapsto \tau * b \in k\mathfrak{t}_{reg} - \rho \tag{5.63a}$$

[36] one can check easily that the final result for $\mathrm{WLO}_{rig}^{disc}(L)$ does not depend on this assumption

[37] again one can check easily that the final result for $\mathrm{WLO}_{rig}^{disc}(L)$ does not depend on this assumption

$$W \times (kP - \rho) \ni (\tau, b) \mapsto \tau * b \in k(Q \cap \mathfrak{t}_{reg}) - \rho \tag{5.63b}$$

where $*$ is given as in Eq. (5.5) in Sec. 5.2 above.

Observe that for $\eta \in \Lambda$ and $\tau \in \mathcal{W}_{\text{aff}}$ we have

$$\theta_{\tau * \eta} = \theta_\eta \tag{5.64a}$$

$$d_{\tau * \eta} = (-1)^\tau d_\eta \tag{5.64b}$$

[Since $\mathcal{W}_{\text{aff}}$ is generated by $\mathcal{W}$ and the translations associated to the lattice Γ it is enough to check Eq. (5.64b) and Eq. (5.64a) for elements of $\mathcal{W}$ and the aforementioned translations. If $\tau \in \mathcal{W}$ then $\tau * \eta = \tau \cdot \eta + \tau \cdot \rho - \rho$. On the other hand if τ is the translation by $y \in \Gamma$ we have $\tau * \eta = \eta + ky$. Using this[38] Eq. (5.64a) follows from Eq. (5.3a) and Eq. (5.64b) follows from Eq. (5.3b) and Eq. (5.1b)].

In the special case where $\tau \in \mathcal{W}$ we also have $\theta^{\frac{\mathbf{q}}{\mathbf{p}}}_{\tau * \eta} = \theta^{\frac{\mathbf{q}}{\mathbf{p}}}_\eta$. However, if $\mathbf{p} \neq \pm 1$ we cannot expect the last relation to hold for a general element τ of $\mathcal{W}_{\text{aff}}$ (cf. Footnote 25 in Sec. 5.2 above).

On the other hand, by taking into account[39] Eq. (5.58e) above and[40] Eq. (5.4) above we see that we always have

$$m_{\lambda_1}\big(\tfrac{1}{\mathbf{p}}(\tau_1 * \eta_1 - \tau_1 * \eta_2)\big)\theta^{\frac{\mathbf{q}}{\mathbf{p}}}_{\tau_1 * \eta_1}\theta^{-\frac{\mathbf{q}}{\mathbf{p}}}_{\tau_1 * \eta_2} = m_{\lambda_1}\big(\tfrac{1}{\mathbf{p}}(\eta_1 - \eta_2)\big)\theta^{\frac{\mathbf{q}}{\mathbf{p}}}_{\eta_1}\theta^{-\frac{\mathbf{q}}{\mathbf{p}}}_{\eta_2} \tag{5.65}$$

for all $\tau_1 \in \mathcal{W}_{\text{aff}}$ and $\eta_1, \eta_2 \in \Lambda$. Combining this with Eq. (5.60) we finally obtain

$$\begin{aligned}
&\mathrm{WLO}^{disc}_{rig}(L) \\
&\sim \sum_{\eta_1,\eta_2 \in (kP-\rho)\cap\Lambda} \sum_{\tau_1 \in W, \tau_2 \in \mathcal{W}_{\text{aff}}} \Big[m_{\lambda_1}\big(\tfrac{1}{\mathbf{p}}(\tau_1 * \eta_1 - \tau_2 * \eta_2)\big) d_{\tau_1 * \eta_1} d_{\tau_2 * \eta_2} \theta^{\frac{\mathbf{q}}{\mathbf{p}}}_{\tau_1 * \eta_1}\theta^{-\frac{\mathbf{q}}{\mathbf{p}}}_{\tau_2 * \eta_2} \Big] \\
&= \sum_{\eta_1,\eta_2 \in (kP-\rho)\cap\Lambda} \sum_{\tau_1 \in W, \tau_2 \in \mathcal{W}_{\text{aff}}} \Big[m_{\lambda_1}\big(\tfrac{1}{\mathbf{p}}(\tau_1 * \eta_1 - \tau_2 * \eta_2)\big) (-1)^{\tau_1}(-1)^{\tau_2} d_{\eta_1} d_{\eta_2} \theta^{\frac{\mathbf{q}}{\mathbf{p}}}_{\tau_1 * \eta_1}\theta^{-\frac{\mathbf{q}}{\mathbf{p}}}_{\tau_2 * \eta_2} \Big] \\
&\overset{(*)}{=} \sum_{\eta_1,\eta_2 \in (kP-\rho)\cap\Lambda} \sum_{\tau_1 \in W, \tau \in \mathcal{W}_{\text{aff}}} \Big[m_{\lambda_1}\big(\tfrac{1}{\mathbf{p}}(\eta_1 - \tau * \eta_2)\big) (-1)^{\tau} d_{\eta_1} d_{\eta_2} \theta^{\frac{\mathbf{q}}{\mathbf{p}}}_{\eta_1}\theta^{-\frac{\mathbf{q}}{\mathbf{p}}}_{\tau * \eta_2} \Big] \\
&\overset{(**)}{\sim} \sum_{\eta_1,\eta_2 \in \Lambda^k_+} \sum_{\tau \in \mathcal{W}_{\text{aff}}} \Big[(-1)^{\tau} m_{\lambda_1}\big(\tfrac{1}{\mathbf{p}}(\eta_1 - \tau * \eta_2)\big) d_{\eta_1} d_{\eta_2} \theta^{\frac{\mathbf{q}}{\mathbf{p}}}_{\eta_1}\theta^{-\frac{\mathbf{q}}{\mathbf{p}}}_{\tau * \eta_2} \Big] \\
&= \sum_{\eta_1,\eta_2 \in \Lambda^k_+} \sum_{\tau \in \mathcal{W}_{\text{aff}}} m^{\eta_1 \eta_2}_{\lambda_1, \mathbf{p}}(\tau) d_{\eta_1} d_{\eta_2} \theta^{\frac{\mathbf{q}}{\mathbf{p}}}_{\eta_1}\theta^{-\frac{\mathbf{q}}{\mathbf{p}}}_{\tau * \eta_2}
\end{aligned} \tag{5.66}$$

where in step $(*)$ we applied Eq. (5.65) and made the change of variable $\tau_2 \to \tau := \tau_1^{-1}\tau_2$ and where in step $(**)$ we have used that (cf. (5.61) above)

$$\begin{aligned}
(kP - \rho) &= \{kb - \rho \mid b \in \mathcal{C} \text{ and } \langle b, \theta \rangle < 1\} \\
&= \{\bar{b} \in \mathfrak{t} \mid \bar{b} + \rho \in \mathcal{C} \text{ and } \langle \bar{b} + \rho, \theta \rangle < k\}
\end{aligned}$$

and therefore

$$\begin{aligned}
\Lambda \cap (kP - \rho) &= \{\lambda \in \Lambda \mid \lambda + \rho \in \mathcal{C} \text{ and } \langle \lambda + \rho, \theta \rangle < k\} \\
&\overset{(*)}{=} \{\lambda \in \Lambda \cap \overline{\mathcal{C}} \mid \langle \lambda + \rho, \theta \rangle < k\} = \Lambda^k_+
\end{aligned}$$

where step $(*)$ follows because for each $\lambda \in \Lambda$, $\lambda + \rho$ is in the open Weyl chamber $\mathcal{C}$ iff λ is in the closure $\overline{\mathcal{C}}$ (cf. the last remark in Sec. V.4 in [10]).

[38] and taking into account the relations $\rho \in \Lambda$, $\forall x, y \in \Gamma : \langle x, y \rangle \in \mathbb{Z}$, and $\forall x \in \Gamma : \langle x, x \rangle \in 2\mathbb{Z}$ (cf. Eq. (5.49) above and the paragraph following Eq. (5.49)

[39] this is relevant only in the special case where τ_1 is a translation by $y \in \Gamma$. If $\tau_1 \in \mathcal{W}$ then the validity of Eq. (5.65) follows from the $\mathcal{W}$-invariance of $m_{\lambda_1}(\cdot)$ and the relations mentioned above

[40] recall that – according to the conventions made above we have $m_{\lambda_1}(\cdot) = \bar{m}_{\lambda_1}(\cdot)$

f) Step 6: Final step

In Steps 1–5 we showed that for L as in Theorem 5.7 we have

$$\mathrm{WLO}^{disc}_{rig}(L) \sim \sum\nolimits_{\eta_1,\eta_2 \in \Lambda^k_+} \sum\nolimits_{\tau \in \mathcal{W}_{\mathrm{aff}}} m^{\eta_1 \eta_2}_{\lambda_1,\mathbf{p}}(\tau)\ d_{\eta_1} d_{\eta_2}\ \theta_{\eta_1}^{\frac{\mathbf{q}}{\mathbf{p}}} \theta_{\tau * \eta_2}^{-\frac{\mathbf{q}}{\mathbf{p}}} \tag{5.67}$$

For the empty simplicial ribbon link $L = \emptyset$ the computations in Step 1–5 simplify drastically and we obtain

$$\mathrm{WLO}^{disc}_{rig}(\emptyset) \sim \frac{1}{S^2_{00}} \tag{5.68}$$

where the multiplicative (non-zero) constant represented by $\sim$ is the same as that in Eq. (5.67) above. Combining Eq. (5.67) and Eq. (5.68) and recalling the meaning of $\sim$ we conclude

$$\mathrm{WLO}^{disc}_{norm}(L) = \frac{\mathrm{WLO}^{disc}_{rig}(L)}{\mathrm{WLO}^{disc}_{rig}(\emptyset)} = S^2_{00} \sum\nolimits_{\eta_1,\eta_2 \in \Lambda^k_+} \sum\nolimits_{\tau \in \mathcal{W}_{\mathrm{aff}}} m^{\eta_1 \eta_2}_{\lambda_1,\mathbf{p}}(\tau)\ d_{\eta_1} d_{\eta_2}\ \theta_{\eta_1}^{\frac{\mathbf{q}}{\mathbf{p}}} \theta_{\tau * \eta_2}^{-\frac{\mathbf{q}}{\mathbf{p}}}$$

Remark 5.14 An alternative (and more explicit) derivation of Eq. (5.68) can be obtained as follows. We can apply Eq. (5.67) to the special situation $L = L' := (R'_1)$ where R'_1 is a simplicial torus ribbon knot in $\mathcal{K} \times \mathbb{Z}_N$ of standard type with winding numbers $\mathbf{p} = 1$ and $\mathbf{q} = 0$ and which is a colored with the trivial representation ρ_0. Since $\mathrm{WLO}^{disc}_{rig}(\emptyset) = \mathrm{WLO}^{disc}_{rig}(L')$ we obtain

$$\begin{aligned}\mathrm{WLO}^{disc}_{rig}(\emptyset) = \mathrm{WLO}^{disc}_{rig}(L') &\sim \sum\nolimits_{\eta_1,\eta_2 \in \Lambda^k_+} \sum\nolimits_{\tau \in \mathcal{W}_{\mathrm{aff}}} m^{\eta_1 \eta_2}_{0,1}(\tau)\ d_{\eta_1} d_{\eta_2}\ \theta_{\eta_1}^{\frac{0}{1}} \theta_{\tau * \eta_2}^{-\frac{0}{1}} \\ &\overset{(*)}{=} \sum\nolimits_{\eta_1 \in \Lambda^k_+} d^2_{\eta_1} = \sum\nolimits_{\eta_1 \in \Lambda^k_+} \tfrac{S_{\eta_1 0} S_{\eta_1 0}}{S_{00} S_{00}} = C_{00} \tfrac{1}{S^2_{00}} = \delta_{0\bar{0}} \tfrac{1}{S^2_{00}} = \tfrac{1}{S^2_{00}}\end{aligned} \tag{5.69}$$

where in step $(*)$ we used $m_0(\alpha) = \delta_{0\alpha}$ which implies that $m^{\eta_1 \eta_2}_{0,1}(\tau)$ vanishes unless $\tau = 1$ and $\eta_2 = \eta_1$.

5.5 Proof of Theorem 5.8

We will now sketch how a proof of Theorem 5.8 can be obtained by a straightforward modification of the proof of Theorem 5.7.

Recall that R_2 appearing in Theorem 5.8 is a closed simplicial ribbon which is vertical and has standard orientation. Let $\sigma^j_2 \in \mathfrak{F}_0(q\mathcal{K})$, $j = 0, 1, 2$, be the three points associated to R_2 as explained in Definition 5.5. We then have

$$\mathrm{Tr}_{\rho_2}\big(\mathrm{Hol}^{disc}_{R_2}(\check{A}^\perp + A^\perp_c, B)\big) = \mathrm{Tr}_{\rho_2}\big(\exp(\mathbf{q}_2 \sum\nolimits_{j=0}^2 w(j) B(\sigma^j_2))\big) \tag{5.70}$$

where $w(j)$, $j = 0, 1, 2$, is as in Sec. 3.5 and where $\mathbf{q}_2$ is the second winding number of R_2. Since R_2 has – by assumption – standard orientation (cf. again Definition 5.5) we have $\mathbf{q}_2 = 1$.

Clearly, the last expression in Eq. (5.70) above does not depend on $\check{A}^\perp$ and $A^\perp_c$. Thus when proving Theorem 5.8 we can repeat the Steps 1–3 in the proof of Theorem 5.7 almost without modifications, the only difference being that now an extra factor $\mathrm{Tr}_{\rho_2}\big(\exp(\sum_{j=0}^2 w(j) B(\sigma^j_2))\big)$ appears in several equations. For example, we obtain again Eq. (5.46) at the end of Step 3 where this time the function $F_\alpha(b)$ contains an extra factor $\mathrm{Tr}_{\rho_2}\big(\exp(\sum_{j=0}^2 w(j) B(\sigma^j_2)))\big)$ inside the $[\cdots]$ brackets. According to Observation 5.12 in Step 3 for all B appearing in $F_\alpha(b)$ we have $B(\sigma^0_2) = B(\sigma^1_2) = B(\sigma^2_2)$, which implies that $\mathrm{Tr}_{\rho_2}\big(\exp(\sum_{j=0}^2 w(j) B(\sigma^j_2))\big) = \mathrm{Tr}_{\rho_2}\big(\exp(B(\sigma^0_2))\big)$ for the relevant B. This extra factor appears later, in Eq. (5.55) (and in the modification Eq. (5.56)) at the end of Step 4. (In order to arrive at the modified version of Eq. (5.55) we need to add the equation $\mathrm{Tr}_{\rho_2}\big(\exp(b + x)\big) = \mathrm{Tr}_{\rho_2}\big(\exp(b)\big)$ for all $b \in \mathfrak{t}$, $x \in I$ to the list of equations in Eqs (5.48)).

As in Sec. 5.4 above let us assume again (without loss of generality) that the enumeration of Z_1 and Z_2 was chosen such that in Eq. (5.37) in the proof of Lemma 5.11 we have the situation where the "$-$"-sign appears. Then it follows from our assumption that R_1 winds around R_2 in "positive direction"[41] that $\sigma_2^0 \in Z_2$. So if we now, in Step 5 replace the variable B by $\eta : \{1,2\} \to \Lambda$ given by $\eta(i) = kB(Z_i) - \rho$ then $B(\sigma_2^0)$ gets replaced by $\frac{1}{k}(\eta(2) + \rho)$ and the term $\mathrm{Tr}_{\rho_2}\big(\exp(B(\sigma_2^0))\big)$ is replaced by $\mathrm{Tr}_{\rho_2}\big(\exp(\frac{1}{k}(\eta(2)+\rho))\big)$. From Weyl's character formula it follows that

$$\mathrm{Tr}_{\rho_2}\big(\exp(\tfrac{1}{k}(\eta(2)+\rho))\big) = \frac{S_{\lambda_2\eta(2)}}{S_{0\eta(2)}}$$

where λ_2 is the highest weight of ρ_2.

Accordingly, at a later stage in Step 5 an extra factor $\frac{S_{\lambda_2\eta_2}}{S_{0\eta_2}}$ appears in several equations where η_2 is one of the two summation variables. Taking into account that apart from Eq. (5.64) we also have $S_{\lambda_2(\tau*\eta)} = (-1)^\tau S_{\lambda_2\eta}$ we finally arrive at

$$\begin{aligned}
\mathrm{WLO}^{disc}_{rig}(L) &\sim \sum_{\eta_1,\eta_2\in\Lambda^k_+} \sum_{\tau\in\mathcal{W}_{\mathrm{aff}}} m^{\eta_1\eta_2}_{\lambda_1,\mathbf{p}}(\tau) d_{\eta_1} d_{\eta_2} \frac{S_{\lambda_2\eta_2}}{S_{0\eta_2}} \theta_{\eta_1}^{\frac{\mathfrak{q}}{\mathbf{p}}} \theta_{\tau*\eta_2}^{-\frac{\mathfrak{q}}{\mathbf{p}}}\\
&= \sum_{\eta_1,\eta_2\in\Lambda^k_+} \sum_{\tau\in\mathcal{W}_{\mathrm{aff}}} m^{\eta_1\eta_2}_{\lambda_1,\mathbf{p}}(\tau) d_{\eta_1} \tfrac{1}{S_{00}} S_{\lambda_2\eta_2} \theta_{\eta_1}^{\frac{\mathfrak{q}}{\mathbf{p}}} \theta_{\tau*\eta_2}^{-\frac{\mathfrak{q}}{\mathbf{p}}} \qquad (5.71)
\end{aligned}$$

Combining this with Eq. (5.68) above (where $\sim$ represents equality up to the same multiplicative constant as in Eq. (5.71)) we arrive at Eq. (5.9).

6 Comparison of $\mathrm{WLO}^{disc}_{norm}(L)$ with the Reshetikhin-Turaev invariant $RT(L)$

6.1 Conjecture 1

For every $\mathfrak{g}$ as in Sec. 2, every $k \in \mathbb{N}$ with $k > c_{\mathfrak{g}}$, and every colored ribbon link L in $S^2 \times S^1$ let us denote by $RT(S^2 \times S^1, L)$, or simply by $RT(L)$, the corresponding Reshetikhin-Turaev invariant associated to $U_q(\mathfrak{g}_{\mathbb{C}})$ where $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ and $q = e^{2\pi i/k}$ (cf. Remark 5.6 above).

According to Remark 3.11 above it is plausible[42] to expect that the values of $\mathrm{WLO}^{disc}_{norm}(L)$ computed above coincide with the corresponding values of Witten's heuristic path integral expressions[43] $\mathrm{WLO}_{norm}(L) = Z(S^2 \times S^1, L)/Z(S^2 \times S^1) = Z(S^2 \times S^1, L)$ in the special case considered. In view of the expected equivalence between Witten's heuristic path integral expressions $Z(S^2 \times S^1, L)$ and the rigorously defined Reshetikhin-Turaev invariants $RT(L)$ we arrive at the following (rigorous) conjecture:

Conjecture 1 *For every colored simplicial ribbon link L as in Theorem 5.7 or Theorem 5.8 we have*

$$\mathrm{WLO}^{disc}_{norm}(L) = RT(L)$$

where on the RHS we consider L as a continuum ribbon link in $S^2 \times S^1$ in the obvious way.

As mentioned already in Remark 5.9 above in the special case $\mathbf{p} = 1$ Conjecture 1 is true. Since I have not found a concrete formula for $RT(L)$ where L is as in Theorem 5.7 or Theorem 5.8 with $\mathbf{p} > 1$ in the standard literature at present I cannot prove Conjecture 1 in general (cf. also Sec. 6.3 below). We can, however, make a "consistency check" and compute – assuming

[41] by which we meant that the winding number of the projected ribbon $(R_1)_\Sigma := \pi_\Sigma \circ R_1 : S^1 \times [0,1] \to S^2$ around σ_2^0 is positive

[42] in view of Footnote 18 and Footnote 20 in Remark 3.11 above we do not have the guarantee that this really is the case

[43] Here we used Witten's heuristic argument that $Z(S^2 \times S^1) = 1$

the validity of Conjecture 1 – the value of $RT(S^3, \tilde{L})$ for an arbitrary colored torus knot $\tilde{L}$ in S^3. We will do this in Sec. 6.2 below with the help of a standard surgery argument. It turns out that we indeed obtain the correct value for $RT(S^3, \tilde{L})$ (which is given by the Rosso-Jones formula, cf. Eq. (6.12) below).

6.2 Derivation of the Rosso-Jones formula

Let us now combine Theorem 5.8 with a simple surgery argument in order to derive[44] the Rosso-Jones formula for general colored torus knots in S^3.

In the following it will be convenient to switch forth and back between the framed link picture and the ribbon link picture.

Let us first recall Witten's (heuristic) surgery formula. For our purposes it will be sufficient to consider the following special case of Witten's surgery formula[45]

$$Z(S^3, \tilde{L}) = \sum\nolimits_{\alpha \in \Lambda^k_+} K_{\alpha 0} \; Z(S^2 \times S^1, L, (C, \rho_\alpha)) \tag{6.1}$$

where

- L is a colored, framed link in $S^2 \times S^1$,
- $\tilde{L}$ is the colored, framed link in S^3 obtained from L by performing a surgery on a separate (framed) knot C in $S^2 \times S^1$,
- ρ_α is the irreducible, finite-dimensional, complex representation of G with highest weight $\alpha \in \Lambda^k_+$ (we assume that C is colored with ρ_α),
- $(K_{\mu\nu})_{\mu,\nu \in \Lambda^k_+}$ is the matrix associated to the surgery mentioned above.

Let us now restrict to the special case where L is the colored knot $L = (T_{\mathbf{p},\mathbf{q}}, \rho_\lambda)$ where $\lambda \in \Lambda^k_+$ and where $T_{\mathbf{p},\mathbf{q}}$ is a torus knot of standard type in $S^2 \times S^1$ with winding numbers $\mathbf{p} \in \mathbb{Z}\backslash\{0\}$ and $\mathbf{q} \in \mathbb{Z}$ (cf. Definition 5.1 and Definition 5.3) and equipped with a "horizontal" framing[46], i.e. a normal vector field on $T_{\mathbf{p},\mathbf{q}}$ which is parallel to the S^2-component of $S^2 \times S^1$. Moreover, let C be a vertical loop in $S^2 \times S^1$ (equipped with a horizontal framing). Let $\tilde{T}_{\mathbf{p},\mathbf{q}}$ be the framed torus knot in S^3 which is obtained from $T_{\mathbf{p},\mathbf{q}}$ by performing the surgery on C which transforms $S^2 \times S^1$ into S^3 and has the matrix $K = S$ associated to it (cf. p. 389 in [43]).

Remark 6.1 Note that up to equivalence and a change of framing every framed torus knot in S^3 can be obtained in this way.

In the special situation described above formula (6.1) reads

$$Z(S^3, (\tilde{T}_{\mathbf{p},\mathbf{q}}, \rho_\lambda)) = \sum\nolimits_{\alpha \in \Lambda^k_+} S_{\alpha 0} \; Z(S^2 \times S^1, (T_{\mathbf{p},\mathbf{q}}, \rho_\lambda), (C, \rho_\alpha)) \tag{6.2}$$

Clearly, Eq. (6.2) is not rigorous. We can obtain a rigorous version of Eq. (6.2) by replacing the two heuristic path integral expressions $Z(S^3, (\tilde{T}_{\mathbf{p},\mathbf{q}}, \rho_\lambda))$ and $Z(S^2 \times S^1, (T_{\mathbf{p},\mathbf{q}}, \rho_\lambda), (C, \rho_\alpha))$ with the corresponding Reshetikhin-Turaev invariants. Doing so we arrive at

$$RT(S^3, (\tilde{T}_{\mathbf{p},\mathbf{q}}, \rho_\lambda)) = \sum\nolimits_{\alpha \in \Lambda^k_+} S_{\alpha 0} \; RT(S^2 \times S^1, (T_{\mathbf{p},\mathbf{q}}, \rho_\lambda), (C, \rho_\alpha)) \tag{6.3}$$

[44] As explained in Remark 6.2 below, we will do this in two different ways. Firstly, in a rigorous way in order to obtain a consistency check of Conjecture 1 above and, secondly, in a heuristic way (where we do not need Conjecture 1) in order to obtain a heuristic derivation of the Rosso-Jones formula

[45] here we use a notation which is very similar to Witten's notation; one important difference is that we write (C, ρ_α) where Witten writes R_α

[46] a special case of $T_{\mathbf{p},\mathbf{q}}$ is any simplicial torus ribbon knot of standard type (cf. Definition 5.4) when considered as a framed knot instead of a ribbon knot

Remark 6.2 Even though Eq. (6.2) is only heuristic it is sufficient/appropriate for achieving "Goal 2" of Comment 1 in the Introduction. It is straightforward to rewrite the next paragraphs using Eq. (6.2) instead of Eq. (6.3) and using[47] $Z(S^2 \times S^1, (T_{\mathbf{p},\mathbf{q}}, \rho_\lambda), (C, \rho_\alpha)) = \mathrm{WLO}_{norm}(L) = \mathrm{WLO}^{disc}_{norm}(L)$ where L is given as in the paragraph after the present remark. Doing so we arrive at

$$Z(S^3, (\tilde{T}_{\mathbf{p},\mathbf{q}}, \rho_\lambda)) = S_{00} \sum\nolimits_{\mu \in \Lambda_+} c^\mu_{\lambda,\mathbf{p}} d_\mu \theta_\mu^{\frac{\mathbf{q}}{\mathbf{p}}}, \tag{6.4}$$

which is the (heuristic) "Chern-Simons path integral version" of the Rosso-Jones formula, cf. Eq. (6.12) below.

On the other hand, for "Goal 1" we should use the rigorous formula Eq. (6.3) in order to make the aforementioned "consistency check" of Conjecture 1 above. Since Goal 1 is our main goal we will now work with Eq. (6.3).

Let us now consider the special case where $T_{\mathbf{p},\mathbf{q}}$ "comes from"[48] a simplicial torus ribbon knot R_1 in $\mathcal{K} \times \mathbb{Z}$ of standard type and C "comes from" a vertical closed simplicial ribbon R_2 in $\mathcal{K} \times \mathbb{Z}$. Moreover, set $\rho_1 := \rho_\lambda$ and $\rho_2 := \rho_\alpha$ where $\alpha \in \Lambda^k_+$ is fixed (temporarily). Finally, assume that for the colored simplicial ribbon link $L := ((R_1, R_2), (\rho_1, \rho_2))$ in $\mathcal{K} \times \mathbb{Z}$ the assumptions of Theorem 5.8 above are fulfilled. If Conjecture 1 is true we have

$$RT(S^2 \times S^1, (T_{\mathbf{p},\mathbf{q}}, \rho_\lambda), (C, \rho_\alpha)) = \mathrm{WLO}^{disc}_{norm}(L) \tag{6.5}$$

Combining Eq. (6.5) with Theorem 5.8 (for every $\alpha \in \Lambda^k_+$) and using Eq. (6.3) we obtain

$$\begin{aligned} RT(S^3, (\tilde{T}_{\mathbf{p},\mathbf{q}}, \rho_\lambda)) &= \sum\nolimits_{\alpha \in \Lambda^k_+} S_{\alpha 0} \Big(S_{00} \sum\nolimits_{\eta_1,\eta_2 \in \Lambda^k_+} \sum\nolimits_{\tau \in \mathcal{W}_{\mathrm{aff}}} m^{\eta_1\eta_2}_{\lambda,\mathbf{p}}(\tau) d_{\eta_1} S_{\alpha\eta_2} \theta_{\eta_1}^{\frac{\mathbf{q}}{\mathbf{p}}} \theta_{\tau*\eta_2}^{-\frac{\mathbf{q}}{\mathbf{p}}} \Big) \\ &= S_{00} \sum\nolimits_{\eta_1,\eta_2 \in \Lambda^k_+} \sum\nolimits_{\tau \in \mathcal{W}_{\mathrm{aff}}} \Big(\sum\nolimits_{\alpha \in \Lambda^k_+} S_{\alpha 0} S_{\alpha\eta_2} \Big) m^{\eta_1\eta_2}_{\lambda,\mathbf{p}}(\tau) d_{\eta_1} \theta_{\eta_1}^{\frac{\mathbf{q}}{\mathbf{p}}} \theta_{\tau*\eta_2}^{-\frac{\mathbf{q}}{\mathbf{p}}} \\ &\overset{(*)}{=} S_{00} \sum\nolimits_{\eta_1,\eta_2 \in \Lambda^k_+} \sum\nolimits_{\tau \in \mathcal{W}_{\mathrm{aff}}} (C_{0\eta_2}) m^{\eta_1\eta_2}_{\lambda,\mathbf{p}}(\tau) d_{\eta_1} \theta_{\eta_1}^{\frac{\mathbf{q}}{\mathbf{p}}} \theta_{\tau*\eta_2}^{-\frac{\mathbf{q}}{\mathbf{p}}} \\ &\overset{(**)}{=} S_{00} \sum\nolimits_{\eta_1 \in \Lambda^k_+} \sum\nolimits_{\tau \in \mathcal{W}_{\mathrm{aff}}} m^{\eta_1 0}_{\lambda,\mathbf{p}}(\tau) d_{\eta_1} \theta_{\eta_1}^{\frac{\mathbf{q}}{\mathbf{p}}} \theta_{\tau*0}^{-\frac{\mathbf{q}}{\mathbf{p}}} \end{aligned} \tag{6.6}$$

Here in Step $(*)$ we used $S^2 = C$ and the fact that S is a symmetric matrix (cf. Sec. 5.2), and in Step $(**)$ we used $C_{0\mu} = \delta_{\bar{0}\mu} = \delta_{0\mu}$.

By renaming the index η_1 as μ we obtain from Eq. (6.6)

$$RT(S^3, (\tilde{T}_{\mathbf{p},\mathbf{q}}, \rho_\lambda)) = S_{00} \sum\nolimits_{\mu \in \Lambda^k_+} \sum\nolimits_{\tau \in \mathcal{W}_{\mathrm{aff}}} m^{\mu 0}_{\lambda,\mathbf{p}}(\tau) d_\mu \theta_\mu^{\frac{\mathbf{q}}{\mathbf{p}}} \theta_{\tau*0}^{-\frac{\mathbf{q}}{\mathbf{p}}} \tag{6.7}$$

For simplicity we will now assume that k is "large" (cf. Remark 6.3 below for the case of general $k > c_{\mathfrak{g}}$). If k is "large enough"[49] the sum $\sum_{\tau \in \mathcal{W}_{\mathrm{aff}}} \cdots$ appearing in Eq. (6.7) can be replaced by $\sum_{\tau \in \mathcal{W}} \cdots$. On the other hand, for $\tau \in \mathcal{W}$ we do have $\theta_{\tau*0}^{-\frac{\mathbf{q}}{\mathbf{p}}} = \theta_0^{-\frac{\mathbf{q}}{\mathbf{p}}} = 1$ and so Eq. (6.7) simplifies and we obtain

$$\begin{aligned} RT(S^3, (\tilde{T}_{\mathbf{p},\mathbf{q}}, \rho_\lambda)) &= S_{00} \sum\nolimits_{\mu \in \Lambda^k_+} \sum\nolimits_{\tau \in \mathcal{W}} m^{\mu 0}_{\lambda,\mathbf{p}}(\tau) d_\mu \theta_\mu^{\frac{\mathbf{q}}{\mathbf{p}}} \\ &= S_{00} \sum\nolimits_{\mu \in \Lambda^k_+} \bar{M}^{\mu 0}_{\lambda,\mathbf{p}} d_\mu \theta_\mu^{\frac{\mathbf{q}}{\mathbf{p}}} \end{aligned} \tag{6.8}$$

[47] cf. the argument at the beginning of Sec. 6.1 where we used Witten's heuristic equation $Z(S^2 \times S^1) = 1$, cf. Footnote 43 above.

[48] i.e. $T_{\mathbf{p},\mathbf{q}}$ agrees with R_1 when R_1 is considered as a framed knot instead of a ribbon knot

[49] i.e. $k \geq k(\lambda, \mathbf{p})$ where $k(\lambda, \mathbf{p})$ is a constant depending only on λ and $\mathbf{p}$

where we have set

$$\bar{M}^{\mu 0}_{\lambda,\mathbf{p}} := \sum\nolimits_{\tau \in \mathcal{W}} m^{\mu 0}_{\lambda,\mathbf{p}}(\tau) = \sum\nolimits_{\tau \in \mathcal{W}} (-1)^{\tau} m_{\lambda}\big(\tfrac{1}{\mathbf{p}}(\mu - \tau \cdot \rho + \rho)\big) \tag{6.9}$$

Observe that (for fixed λ and $\mathbf{p}$) the coefficients $\bar{M}^{\mu 0}_{\lambda,\mathbf{p}}$ are non-zero only for a finite number of values of μ. So if k is large enough we can replace the index set Λ^k_+ in Eq. (6.8) by Λ_+ and obtain

$$RT(S^3, (\tilde{T}_{\mathbf{p},\mathbf{q}}, \rho_\lambda)) = S_{00} \sum\nolimits_{\mu \in \Lambda_+} \bar{M}^{\mu 0}_{\lambda,\mathbf{p}} d_\mu \theta_\mu^{\frac{\mathbf{q}}{\mathbf{p}}} \tag{6.10}$$

Remark 6.3 For simplicity, we considered here (and in the paragraph before Eq. (6.8)) the case where k is "large". However, it is not too difficult to see that this restriction on k can be dropped, i.e. assuming the validity of Conjecture 1 we can actually derive Eq. (6.10) for all $k > c_{\mathfrak{g}}$.

According to Lemma 2.1 in [13] we have

$$\forall \mu, \lambda \in \Lambda_+ : \forall \mathbf{p} \in \mathbb{N} : \quad \bar{M}^{\mu 0}_{\lambda,\mathbf{p}} = c^{\mu}_{\lambda,\mathbf{p}} \tag{6.11}$$

where $(c^{\mu}_{\lambda,\mathbf{p}})_{\mu,\lambda \in \Lambda_+, \mathbf{p} \in \mathbb{N}}$ are the "plethysm coefficients" appearing in the Rosso-Jones formula, cf. [35] and Eq. (10) in [12]. Accordingly, we can rewrite Eq. (6.10) as

$$RT(S^3, (\tilde{T}_{\mathbf{p},\mathbf{q}}, \rho_\lambda)) = S_{00} \sum\nolimits_{\mu \in \Lambda_+} c^{\mu}_{\lambda,\mathbf{p}} d_\mu \theta_\mu^{\frac{\mathbf{q}}{\mathbf{p}}}, \tag{6.12}$$

which is a version of the Rosso-Jones formula. (Note that the original Rosso-Jones formula deals with unframed torus knots rather than framed torus knots. In Appendix A below we will show that Eq. (6.12) above is indeed equivalent to the original Rosso-Jones formula).

6.3 Reformulation of Conjecture 1

Note that Eq. (6.3) above can be generalized to

$$RT(S^3, (\tilde{T}_{\mathbf{p},\mathbf{q}}, \rho_\lambda), (\tilde{C}, \rho_\beta)) = \sum\nolimits_{\alpha \in \Lambda^k_+} S_{\alpha\beta} \; RT(S^2 \times S^1, (T_{\mathbf{p},\mathbf{q}}, \rho_\lambda), (C, \rho_\alpha)) \quad \forall \beta \in \Lambda^k_+ \tag{6.13}$$

where $((\tilde{T}_{\mathbf{p},\mathbf{q}}, \rho_\lambda), (\tilde{C}, \rho_\beta))$ on the LHS is the (framed, colored) two-component-link in S^3 obtained from the two-component-link $((T_{\mathbf{p},\mathbf{q}}, \rho_\lambda), (C, \rho_\alpha))$ in $S^2 \times S^1$ after applying the same surgery operation as the one described in the paragraph before Remark 6.1 in Sec. 6.2 above.

By modifying the arguments and calculations after Remark 6.2 in Sec. 6.2 above in the obvious way we can show that Conjecture 1 implies that for all $\lambda, \beta \in \Lambda^k_+$ we have

$$RT(S^3, (\tilde{T}_{\mathbf{p},\mathbf{q}}, \rho_\lambda), (\tilde{C}, \rho_\beta)) = S_{00} \sum\nolimits_{\mu \in \Lambda^k_+} \sum\nolimits_{\tau \in \mathcal{W}_{\text{aff}}} m^{\mu \bar{\beta}}_{\lambda,\mathbf{p}}(\tau) d_\mu \theta_\mu^{\frac{\mathbf{q}}{\mathbf{p}}} \theta_{\tau * \bar{\beta}}^{-\frac{\mathbf{q}}{\mathbf{p}}} \tag{6.14}$$

The converse is also true: If Eq. (6.14) holds for every (framed) colored two-component-link $((\tilde{T}_{\mathbf{p},\mathbf{q}}, \rho_\lambda), (\tilde{C}, \rho_\beta))$, $\lambda, \beta \in \Lambda^k_+$ in S^3 obtained as above then Conjecture 1 will be true. I expect that Eq. (6.14) (and therefore Conjecture 1) can be proven for arbitrary[50] $\beta \in \Lambda^k_+$ by using similar techniques as the ones used in [35].

[50] note that according to Sec. 6.2 above in the special case where $\beta = 0$ (and $\lambda \in \Lambda^k_+$ is arbitrary) Eq. (6.14) is indeed true.

7 Conclusions

In the present paper we introduced and studied – for every simple, simply-connected, compact Lie group G, and a large class of colored torus (ribbon) knots[51] L in $S^2 \times S^1$ – a rigorous realization $\mathrm{WLO}^{disc}_{norm}(L)$ of the torus gauge fixed version of Witten's heuristic CS path integral expressions $Z(S^2 \times S^1, L)$. Moreover, we computed the values of $\mathrm{WLO}^{disc}_{norm}(L)$ explicitly, cf. Theorem 5.7.

As a by-product we obtained an elementary, heuristic derivation[52] of the original Rosso-Jones formula for arbitrary colored torus knots $\tilde{L}$ in S^3 (and arbitrary simple complex Lie algebras $\mathfrak{g}_{\mathbb{C}}$). This means that we have achieved "Goal 2" of Comment 1 in the Introduction.

Apart from achieving "Goal 2" we have also made progress towards achieving "Goal 1" of Comment 1. The rigorous computation in Sec. 6.2 provides strong evidence in favor of Conjecture 1 above, i.e. the conjecture that the explicit values of $\mathrm{WLO}^{disc}_{norm}(L)$ obtained in Theorem 5.7 and Theorem 5.8 indeed coincide with the values of the corresponding Reshetikhin-Turaev invariants $RT(L)$. If this is indeed the case Theorem 5.7 can be considered as a step forward in the simplicial program for Chern-Simons theory, cf. Sec. 3 of [19] and cf. also Remark 3.10 in Sec. 3 of the present paper.

A Appendix: The original Rosso-Jones formula for unframed torus knots in S^3

In this appendix we will recall the original Rosso-Jones formula (which deals with unframed torus knots in S^3 rather than framed torus knots) and show that it is equivalent to Eq. (6.12) in Sec. 6.2 above.

Recall that above we set $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ and $q = e^{2\pi i/k}$ and denoted by $RT(M, \cdot)$ (for $M = S^3$ and $M = S^2 \times S^1$) the Reshetikhin-Turaev invariant associated to $U_q(\mathfrak{g}_{\mathbb{C}})$. Let us write $RT(\cdot)$ instead of $RT(S^3, \cdot)$.

We will now compare $RT(\cdot)$ with $QI(\cdot)$ where $QI(\cdot)$ is the $U_q(\mathfrak{g}_{\mathbb{C}})$-analogue[53] of the topological invariant for colored links in S^3 which appeared in [40, 35].

The two invariants $RT(\cdot)$ and $QI(\cdot)$ are very closely related but there are some important differences:

- $QI(\cdot)$ is an invariant of (unframed) colored links. More precisely, it is an ambient isotopy invariant (i.e. invariant under[54] Reidemeister I, II and III moves)
- $QI(\cdot)$ is normalized such that $QI(U_\lambda) = 1$ where U_λ is the unknot colored with (the irreducible complex representation ρ with highest weight) $\lambda \in \Lambda^k_+$.

By contrast we have

- $RT(\cdot)$ is an invariant of framed, colored links. More precisely, it is a regular isotopy invariant (i.e. invariant only under Reidemeister II and III moves)

[51] In the present paper we have restricted ourselves to the case of torus knots but it should not be difficult to generalize our main results to torus links.

[52] Our derivation is "almost" a pure path integral derivation. Essentially all our arguments are based on (the rigorous realization of) the Chern-Simons path integral in the torus gauge introduced in Sec. 3. The only two exceptions are the arguments involving Witten's heuristic surgery formula and Witten's formula $Z(S^2 \times S^1) = 1$ mentioned in Remark 6.2 above. (Both formulas were derived by Witten using arguments from Conformal Field Theory.)

[53] Note that in [40, 35] the letter q refers to a complex variable or, equivalently, a generic element of $\mathbb{C}$. We now replace this variable by the root of unity $e^{2\pi i/k}$. Doing so we obtain a complex valued topological invariant of all colored links L in S^3 whose colors ρ_i fulfill the condition $\lambda_i \in \Lambda^k_+$ where λ_i is the highest weight of ρ_i. Recall that if $\lambda_i \notin \Lambda^k_+$ then $RT(L)$ is not defined and $QI(L)$ need not be defined either since division by 0 may occur.

[54] here we consider every link $L \subset S^3$ as a link in $\mathbb{R}^3$ and project it down to a suitable fixed plane P

- For every framed knot L with color ρ_λ, $\lambda \in \Lambda_+^k$, the value of $RT(L)$ changes by a factor $\theta_\lambda^{\pm 1}$ when we perform a Reidemeister I move on L.
- $RT(\cdot)$ is normalized such that $RT(U_\lambda) = S_{\lambda 0}$ where U_λ is the 0-framed unknot colored with λ.

Taking this into account one can deduce the following relation between $RT(\cdot) = RT(S^3, \cdot)$ and $QI(\cdot)$

$$QI(L^0) = \frac{1}{S_{\lambda 0}} \theta_\lambda^{-\operatorname{writhe}(D(L))} RT(S^3, L) \tag{A.1}$$

for every framed knot L with color ρ_λ, $\lambda \in \Lambda_+^k$, where L^0 is the unframed, colored knot obtained from L by forgetting the framing and where $D(L)$ is any "admissible"[55] knot diagram of L. [Note that the writhe is also a regular isotopy invariant and the effect of a Reidemeister I move on the exponential on the RHS of Eq. (A.1) cancels out the effect of the move on the factor $RT(L)$. Accordingly, the RHS of Eq (A.1) will be invariant under Reidemeister I-III moves.]

Let us now consider the special case $L = (\tilde{T}_{\mathbf{p},\mathbf{q}}, \rho_\lambda)$ where $\tilde{T}_{\mathbf{p},\mathbf{q}}$ is the framed torus knot in S^3 that appeared in Sec. 6.2 above. It can be shown that for a torus knot $\tilde{T}_{\mathbf{p},\mathbf{q}}$ obtained by surgery from a torus knot $T_{\mathbf{p},\mathbf{q}}$ in $S^2 \times S^1$ of standard type with horizontal framing we have

$$\operatorname{writhe}(D(\tilde{T}_{\mathbf{p},\mathbf{q}})) = \mathbf{pq}$$

for every admissible knot diagram $D(\tilde{T}_{\mathbf{p},\mathbf{q}})$ of $\tilde{T}_{\mathbf{p},\mathbf{q}}$ (in the sense above). Accordingly, Eq. (A.1) specializes to

$$QI((\tilde{T}^0_{\mathbf{p},\mathbf{q}}, \rho_\lambda)) = \frac{1}{S_{\lambda 0}} \theta_\lambda^{-\mathbf{pq}} RT(S^3, (\tilde{T}_{\mathbf{p},\mathbf{q}}, \rho_\lambda)) \tag{A.2}$$

In view of Eq. (A.2) it is now clear that Eq. (6.12) in Sec. 6.2 above is equivalent to

$$QI((\tilde{T}^0_{\mathbf{p},\mathbf{q}}, \rho_\lambda)) = \frac{1}{d_\lambda} \theta_\lambda^{-\mathbf{pq}} \sum_{\mu \in \Lambda_+} c^\mu_{\lambda,\mathbf{p}} d_\mu \theta_\mu^{\frac{\mathbf{q}}{\mathbf{p}}}, \tag{A.3}$$

which is the original Rosso-Jones formula, cf. Eq. (10) in [12].

References

[1] D. H. Adams. A doubled discretization of abelian Chern-Simons theory. *Phys. Rev. Lett.*, 78(22):4155–4158, 1997.

[2] S. Axelrod and I.M. Singer. Chern-Simons perturbation theory. In Catto, Sultan et al., editor, *Differential geometric methods in theoretical physics. Proceedings of the 20th international conference, June 3-7, 1991, New York City, NY, USA*, volume 1-2, pages 3–45. World Scientific, Singapore, 1992.

[3] S. Axelrod and I.M. Singer. Chern-Simons perturbation theory. II. *J. Diff. Geom.*, 39(1):173–213, 1994.

[4] D. Bar-Natan. Perturbative Chern-Simons theory. *J. Knot Th. Ram.*, 4:503–547, 1995.

[5] C. Beasley and E. Witten. Non-abelian localization for Chern-Simons theory. *J. Diff. Geom.*, 70(2):183–323, 2005.

[6] C. Beasley. Localization for Wilson Loops in Chern-Simons Theory. *Adv. Theor. Math. Phys.*, 17, 1-240, 2013. [arXiv:0911.2687]

[55] Here by "admissible" I mean a knot diagram which is obtained in the following way (here we use the ribbon picture, i.e. we consider the framed knot L as a ribbon in the obvious way): We press the ribbon L flat onto a fixed plane P. After that the ribbon width is sent to zero

[7] M. Blau and G. Thompson. Derivation of the Verlinde Formula from Chern-Simons Theory and the G/G model. *Nucl. Phys. B*, 408(1):345–390, 1993.

[8] M. Blau and G. Thompson. On Diagonalization in $Map(M, G)$. *Commun. Math. Phys.*, 171:639–660, 1995.

[9] R. Bott and C. Taubes. On the self-linking of knots. *J. Math. Phys.*, 35(10):5247–5287, 1994.

[10] Th. Bröcker and T. tom Dieck. *Representations of compact Lie groups*, volume 98 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1985.

[11] S. de Haro and A. Hahn. Chern-Simons theory and the quantum Racah formula. *Rev. Math. Phys.*, 25:1350004, 41 pp., 2013.

[12] S. Garoufalidis, H. Morton, and T. Vuong. The SL_3 colored Jones polynomial of the Trefoil. *Proc. Am. Math. Soc.*, 141(6):2209–2220, 2013. [arXiv:1010.3147]

[13] S. Garoufalidis and T. Vuong. A stability conjecture for the colored Jones polynomial. Preprint, 2013. [arXiv:1310.7143]

[14] E. Guadagnini, M. Martellini, and M. Mintchev. Wilson Lines in Chern-Simons theory and Link invariants. *Nucl. Phys. B*, 330:575–607, 1990.

[15] E. Guadagnini, M. Martellini, and M. Mintchev. Braids and quantum group symmetry in Chern-Simons theory. *Nucl. Phys. B*, 336:581–609, 1990.

[16] A. Hahn. Chern-Simons models on $S^2 \times S^1$, torus gauge fixing, and link invariants I, *J. Geom. Phys.*, 53(3): 275-314, 2005.

[17] A. Hahn. Chern-Simons models on $S^2 \times S^1$, torus gauge fixing, and link invariants II, *J. Geom. Phys.*, 58(9): 1124-1136, 2008.

[18] A. Hahn. An analytic Approach to Turaev's Shadow Invariant. *J. Knot Th. Ram.*, 17(11): 1327–1385, 2008.

[19] A. Hahn. From simplicial Chern-Simons theory to the shadow invariant I, *J. Math. Phys.* 56: 032301, 52 pp., 2015. [arXiv:1206.0439]

[20] A. Hahn. From simplicial Chern-Simons theory to the shadow invariant II, *J. Math. Phys.* 56: 032302, 46 pp., 2015. [arXiv:1206.0441]

[21] A. Hahn. R-matrices and the Chern-Simons path integral. In preparation.

[22] J. Isidro, J. M. F. Labastida, and A. V. Ramallo. Polynomials for torus links from Chern-Simons gauge theory. *Nucl.Phys. B*, 398:187–236, 1993. [arXiv:hep-th/9210124].

[23] V. Jones. Hecke algebra representations of braid groups and link polynomials, *Ann. Math.* 126, 335–388, 1987.

[24] L. Kauffman. *Knots*. World Scientific, Singapore, 1993.

[25] M. Kontsevich. Graphs, homotopical algebra and low-dimensional topology, preprint, 1992.

[26] J. M. F. Labastida, P. M. Llatas, and A. V. Ramallo. Knot operators in Chern-Simons gauge theory, *Nucl. Phys. B* 348:651–692, 1991.

[27] J. M. F. Labastida and M. Mariño. The HOMFLY polynomial for Torus Links from Chern-Simons Gauge Theory. *Int. J. Mod. Phys. A* 10(7):1045–1089, 1994. [hep-th/9402093]

[28] J. M. F. Labastida and M. Mariño. Polynomial Invariants for Torus Knots and Topological Strings. *Comm. Math. Phys.* 217:423–449, 2001. [hep-th/0004196]

[29] J. M. F. Labastida and E. Pérez. A Relation between the Kauffman and the HOMFLY Polynomials for Torus Knots *J. Math. Phys.* 37(4):2013–2042, 1995. [arXiv:q-alg/8507031]

[30] J. M. F. Labastida and A. V. Ramallo. Operator Formalism for Chern-Simons theory, *Phys. Lett. B* 227:92, 1989.

[31] P. Mnëv. Discrete BF theory, [arXiv: 0809.1160v2], 2008

[32] H. Morton. The colored Jones function and Alexander polynomial for torus knots. *Math. Proc. Camb. Phil. Soc.*, 117:129–135, 1995.

[33] N.Y. Reshetikhin and V. G. Turaev. Ribbon graphs and their invariants derived from quantum groups. *Commun. Math. Phys.*, 127:1–26, 1990.

[34] N.Y. Reshetikhin and V. G. Turaev. Invariants of three manifolds via link polynomials and quantum groups. *Invent. Math.*, 103:547–597, 1991.

[35] M. Rosso and V. Jones. On the invariants of torus knots derived from quantum groups. *J. Knot Th. Ram.*, 2:97–112, 1993.

[36] L. Rozansky. A contribution of the trivial connection to Jones polynomial and Witten's invariant of 3d manifolds. I, *Commun. Math. Phys.*, 175:275-296, 1996. [arXiv:hep-th/9401061]

[37] L. Rozansky. A large k asymptotics for Witten's invariant of Seifert manifolds, *Commun. Math. Phys.*, 171:279-322, 1995.

[38] S. Sawin. Quantum groups at roots of unity and modularity, *J. Knot Th. Ram.*, 15(10):1245–1277, 2006.

[39] S. Stevan. Chern-Simons invariants of Torus Links. Annales Henri Poincaré 11:1201–1224, 2010. [arXiv:1003.2861]

[40] V. G. Turaev. The Yang-Baxter equation and invariants of links. *Invent. Math.*, 92(3):527-553, 1988.

[41] V. G. Turaev. Shadow links and face models of statistical mechanics. *J. Diff. Geom.*, 36:35–74, 1992.

[42] V. G. Turaev. *Quantum invariants of knots and 3-manifolds*. De Gruyter, Berlin, 1994.

[43] E. Witten. Quantum Field Theory and the Jones Polynomial. *Commun. Math. Phys.*, 121:351–399, 1989.